Powers of skew-morphisms

Martin Bachratý

joint work with Robert Jajcay

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Skew-morphisms

Let G be a finite group.

A permutation φ of G is a skew-morphism if:

- $\varphi(id_G) = id_G$,
- there exists a (power) function $\pi : G \to \{0, 1, \dots, |\varphi| - 1\}$ such that $\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$, for each $h \in G$.

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In abelian case we have:

$$\varphi(g + h) = \varphi(g) + \varphi^{\pi(g)}(h), \text{ for each } h \in G.$$

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- ◇ automorphisms $\subseteq$ skew-morphisms
- ◇ the structure of automorphisms is much simpler then the structure of skew-morphisms
- ◇ the power of an automorphism is again an automorphism; how about skew-morphisms?

Why study powers?

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Is there a power i such that φ^i is a non-trivial automorphism?
The answer in general is no.

Our goal is to find a power i such that φ^i will be a non-trivial skew-morphism with some special property.

Powers of skew-morphisms

Recall that $\varphi(a+b) = \varphi(a) + \varphi^{\pi(a)}(b)$

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$$\begin{aligned}\varphi^2(a+b) &= \varphi(\varphi(a+b)) = \varphi(\varphi(a) + \varphi^{\pi(a)}(b)) = \\ &= \varphi(\varphi(a)) + \varphi^{\pi(\varphi(a))}(\varphi^{\pi(a)}(b)) = \\ &= \varphi^2(a) + \varphi^{\pi(a)+\pi(\varphi(a))}(b)\end{aligned}$$

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In general $\varphi^i(a+b) = \varphi^i(a) + \varphi^{\pi(a)+\pi(\varphi(a))+\cdots+\pi(\varphi^{i-1}(a))}(b)$

Example

Let φ be the following skew-morphism of Z_{49} of the order 21 (from Conder's list):

$$\varphi = (1, 9, 46, 8, 23, 25, 15, 37, 4, 22, 2, 32, 29, 16, 11, 36, 30, 39, 43, 44, 18)$$
$$(3, 20, 47, 24, 13, 33, 45, 6, 19, 17, 48, 5, 38, 41, 40, 10, 34, 26, 31, 27, 12)$$
$$(7, 14, 28)(21, 42, 35)$$

The power function on orbits:

$$[4, 7, 13, 4, 7, 13, 4, 7, 13, 4, 7, 13, 4, 7, 13, 4, 7, 13, 4, 7, 13]$$
$$[10, 19, 16, 10, 19, 16, 10, 19, 16, \ldots]$$
$$[1, 1, 1][1, 1, 1]$$

Is $\tau = \varphi^2$ a skew-morphism?

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$(1, 9, 46, 8, 23, 25, 15, 37, 4, 22, 2, 32, 29, 16, 11, 36, 30, 39, 43, 44, 18)$
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Suppose that τ is a skew-morphism with the power function $\widetilde{\pi}$.

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$$\begin{aligned}\tau(9+b) &= \varphi^2(9+b) = \varphi^2(9) + \varphi^{\pi(9)+\pi(46)}(b) = \varphi^2(9) + \varphi^{20}(b) \\ &= \tau(9) + \tau^{10}(b), \text{ so } \widetilde{\pi}(9) = 10.\end{aligned}$$

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Useful powers should divide $|\varphi| = 21$: φ^7 and φ^3

The power k_φ

$$\varphi = (1,5,4,2,16,17,10,23,13,20,25,8,19,14,22,11,7,26)$$
$$(3,6,12,24,21,15)(9,18)$$

The power function on orbits:

$$[15,11,3,5,9,17,15,11,3,5,9,17,15,11,3,5,9,17]$$
$$[7,13,7,13,7,13][1,1]$$

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The permutation φ^{k_φ} is a skew-morphism of A.

The results for k_φ

Theorem, B., Jajcay, 2014+

Let A be an abelian group and let φ be a skew-morphism of A with a generating orbit. Then φ^{k_φ} is a non-identity skew-morphism with a constant orbit power function.

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Example: $\varphi = (1, 9, 5, 7, 15, 11, 13, 3, 17)(2, 14, 8)(4, 10, 16)$,
with the power function: $[2, 5, 8, 2, 5, 8, 2, 5, 8][7, 7, 7][4, 4, 4]$

$k_\varphi = 3$

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$\varphi^3 = (1,7,13)(3,9,15)(5,11,17)$,
with the (constant orbit) power function: $[2,2,2][2,2,2][2,2,2]$

The results for cyclic groups

Corollary

Let Z_n be a cyclic group. Then:

- φ^{k_φ} *is always a non-identity skew-morphism with a c.o.p.f.*

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Let Z_n be a cyclic group. Then:

- $\diamond$ φ^{k_φ} *is always a non-identity skew-morphism with a c.o.p.f.*
- $\diamond$ *if* φ *has a generating orbit closed under inverses and* $k_\varphi > 1$, *than* φ^{k_φ} *is either non-identity t-balanced skew-morphism or non-identity automorphism of* Z_n

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- for cyclic groups up to Z_{52} is the number of skew-morphisms with a c.o.p.f. (not counting automorphisms) $\approx$ the number of skew-morphisms without a c.o.p.f.;
- for cyclic groups up to the order 500 there is 177753 skew-morphisms with a c.o.p.f., out of which 76115 are automorphisms (Z_{480} admits 2144 skew-morphisms with a c.o.p.f.)

Thank you!