Locally-transitive graphs and group amalgams

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All graphs are connected and simple. (Not always finite.)
Motivating theorems

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A graph is locally-transitive if every vertex-stabiliser acts transitively on the neighbours of the corresponding vertex. It is easy to see that a locally-transitive graph is edge-transitive and that it is either arc-transitive, or is bipartite and has two orbits on vertices.
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Theorem (Goldschmidt (1980))

In a finite cubic \textit{locally-transitive} graph, arc-stabilisers have order at most 128.
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The Goldschmidt Conjecture

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**Conjecture**

*Let $p$ and $q$ be prime numbers. There exists a constant $c$ such that, in a finite locally-transitive graph with valencies \{p, q\}, arc-stabilisers have order at most $c$.***
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Theorem (Morgan (2013))

In a connected finite 5-valent locally-transitive graph, arc-stabilisers have order at most $5! \cdot 4!^5$. 
Local action

Definition
Let $L_1$ and $L_2$ be finite transitive permutation groups and let $\Gamma$ be a $G$-locally-transitive graph. We say that $(\Gamma, G)$ is locally $[L_1, L_2]$ if, for some edge $\{u, v\}$ of $\Gamma$, we have permutation isomorphisms $G_{\Gamma(u)} \cong L_1$ and $G_{\Gamma(v)} \cong L_2$.

(Trivial) Example: if $L_1$ and $L_2$ are regular then $[L_1, L_2]$ is locally-restrictive.

(Non-trivial) Example: Goldschmidt's result implies that $[Z_3, Z_3]$, $[Z_3, \text{Sym}(3)]$ and $[\text{Sym}(3), \text{Sym}(3)]$ are locally-restrictive.
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Problem
When is $[L_1, L_2]$ locally-restrictive?
Rank two amalgams

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4. The permutation type of the amalgam is $[L_1, L_2]$ where $L_i$ is the permutation group induced by $P_i$ in its action on the right cosets of $B$ in $P_i$. 
Equivalent formulations of the problem

Lemma

Let $L_1$ and $L_2$ be finite transitive permutation groups. The following are equivalent:

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Proof. A bit of Basse-Serre theory + a few tricks.
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3. For every integer $c$, there exists a locally $[L_1, L_2]$ pair $(\mathcal{T}, G)$ such that $\mathcal{T}$ is an infinite tree and $c \leq |G_{uv}| < \infty$ for some edge $\{u, v\}$ of $\mathcal{T}$.

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Proof.
A bit of Basse-Serre theory + a few tricks.
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Semiprimitive groups

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Examples:

1. Regular groups
2. Primitive and quasiprimitive groups
3. Frobenius groups
4. $\text{GL}(V)$ acting on the non-zero elements of $V$
5. $(V \oplus \cdots \oplus V) \rtimes \text{GL}(V)$
Our main theorem

Theorem (Morgan, Spiga, V.)

Let $L_1$ and $L_2$ be finite transitive permutation groups. If one of $L_1$ or $L_2$ is not semiprimitive then $[L_1, L_2]$ is not locally-restrictive.
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Conjecture (Morgan, Spiga, V.)

Let $L_1$ and $L_2$ be finite transitive permutation groups. Then $[L_1, L_2]$ is locally-restrictive if and only if both $L_1$ and $L_2$ are semiprimitive.
Let $k \geq 2$. A rank $k$ amalgam $\mathcal{A}$ is a set of $k$ finite groups $P_1, \ldots, P_k$, such that $\bigcap_{i=1}^{k} P_i \neq \emptyset$ and, for every $i, j \in \{1, \ldots, k\}$ the group operations defined on $P_i$ and $P_j$ coincide when restricted to $P_i \cap P_j$. 


Amalgams of higher rank

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1. $\bigcap_{i=1}^{k} P_i$ is called the Borel subgroup of $\mathcal{A}$ and is denoted $\mathcal{B}(\mathcal{A})$. 
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Amalgams of higher rank II

Theorem (Morgan, Spiga, V.)

Let \( k \geq 3 \) and let \( L_1, \ldots, L_k \) be nontrivial finite transitive permutation groups. The following are equivalent:

1. One of \( L_1, \ldots, L_k \) is not regular.

2. For every integer \( c \), there exists a rank \( k \) faithful amalgam of permutation type \([L_1, \ldots, L_k]\) with Borel subgroup of order at least \( c \).
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2. For every integer $c$, there exists a rank $k$ faithful amalgam of permutation type $[L_1, \ldots, L_k]$ with Borel subgroup of order at least $c$.

This is surprisingly different from the $k = 2$ case!
The end.

Thank you!