Locally-transitive graphs and group amalgams

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Convention

All graphs are connected and simple. (Not always finite.)

Motivating theorems

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Theorem (Goldschmidt (1980))

In a finite cubic locally-transitive graph, arc-stabilisers have order at most 128.

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Theorem (Morgan (2013))

In a connected finite 5-valent locally-transitive graph, arc-stabilisers have order at most $5! \cdot 4!^5$.

Local action

Definition

Let L_1 and L_2 be finite transitive permutation groups and let Γ be a G-locally-transitive graph. We say that (Γ, G) is locally $[L_1, L_2]$ if, for some edge $\{u, v\}$ of Γ, we have permutation isomorphisms $G_u^{\Gamma(u)} \cong L_1$ and $G_v^{\Gamma(v)} \cong L_2$.

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$[L_1, L_2]$ is locally-restrictive if there exists a constant c such that, if Γ is a finite G-locally-transitive graph with (Γ, G) locally $[L_1, L_2]$ and $\{u, v\}$ is an edge of Γ, then $|G_{uv}| \leq c$.

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(Trivial) Example : if L_1 and L_2 are regular then $[L_1, L_2]$ is locally-restrictive.

(Non-trivial) Example : Goldschmidt's result implies that $[\mathbb{Z}_3, \mathbb{Z}_3]$, $[\mathbb{Z}_3, \mathrm{Sym}(3)]$ and $[\mathrm{Sym}(3), \mathrm{Sym}(3)]$ are locally-restrictive.

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Problem

When is $[L_1, L_2]$ locally-restrictive?

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4. The permutation type of the amalgam is $[L_1, L_2]$ where L_i is the permutation group induced by P_i in its action on the right cosets of B in P_i.

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3. *For every integer c, there exists a locally $[L_1, L_2]$ pair $(\mathfrak{T}, G)$ such that $\mathfrak{T}$ is an infinite tree and $c \leq |G_{uv}| < \infty$ for some edge $\{u, v\}$ of $\mathfrak{T}$.*

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Proof.

A bit of Basse-Serre theory + a few tricks. □

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Examples:

1. Regular groups
2. Primitive and quasiprimitive groups
3. Frobenius groups
4. $\mathrm{GL(V)}$ acting on the non-zero elements of V
5. $(V \oplus \cdots \oplus V) \rtimes \mathrm{GL(V)}$

Our main theorem

Theorem (Morgan, Spiga, V.)

Let L_1 and L_2 be finite transitive permutation groups. If one of L_1 or L_2 is not semiprimitive then $[L_1, L_2]$ is not locally-restrictive.

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Conjecture (Morgan, Spiga, V.)

Let L_1 and L_2 be finite transitive permutation groups. Then $[L_1, L_2]$ is locally-restrictive if and only if both L_1 and L_2 are semiprimitive.

Amalgams of higher rank

Let $k \geq 2$. A rank k amalgam $\mathcal{A}$ is a set of k finite groups $P_1, \ldots, P_k$, such that $\bigcap_{i=1}^{k} P_i \neq \emptyset$ and, for every $i, j \in \{1, \ldots, k\}$ the group operations defined on P_i and P_j coincide when restricted to $P_i \cap P_j$.

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Amalgams of higher rank II

Theorem (Morgan, Spiga, V.)

Let $k \geq 3$ and let $L_1, \ldots, L_k$ be nontrivial finite transitive permutation groups. The following are equivalent:

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This is surprisingly different from the $k = 2$ case!

The end.

Thank you!