Symmetry of immersed surfaces in euclidean 3-space

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We will talk briefly about more general immersions later.

History

The question has been considered for embedded surfaces by Rüedy (1971, rotation), TT (Field's Notes to appear, all orientation-preserving), Ko(1993 for bordered surfaces), Costa (1997 anticonformal and 2011 dihedral), Lin (1979, dihedral).

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Only TT does platonic groups. No one does immersed (so all orientable, except bordered).

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Remember that a 3-page book containts a möbius strip so a figure-eight torus does too.

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The equation RH does not classify actions of G on the surface S, but it is a first step.

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For n-fold rotational symmetry we get:

$$\text{RRH} \ : \chi(S) = n\chi(T) - b(n-1)$$

where b is even.

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For even $\chi(T) = -n - 2$, we have
$\pi(T) = \langle x_1, y_1, \cdots x_n, y_n, w, z : \Pi[x_i, y_i]zwz^{-1}w = 1\rangle$
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medskip
Notice that any structure you have on T lifts to S (e.g polyhedral, Riemann surface, smooth), even the nature of the singularities.
This is so much better than building models.

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ON impossible since $q_*(\pi^o(T - B))$ generates $\pi(E^3 - Y)$.

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For ON and even $\chi(T)$, just add orientable handles as needed to Figure-eight klein bottle.For odd $\chi(T)$, $q_*(z^2) = 0$ in infinite cyclic $\pi(E^3/G - Y)$, so $q_*(z) = 0$, making S nonorienatble.

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Since A_4 and A_5 have no index two subgroup, ON is possible only for PRH and CRH.

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Thus number of branch points of each type have same parity (odd if origin inside, even if origin outside).

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Proof Start with sphere intersecting axes correct number of times. Then add orientable handles for OO or crosscaps for NN.
For ON, we have $\pi(E^3 - Y)$ is free of rank two and generated by little loops on $q(T)$ around axes, which are necessarily orientable, so $q_*(\pi^o(T))$ generates $\pi(E^3/G - Y)$ so $\phi(\pi^o(T))$ not index two.

Theorem with no branching

When there is no branching, clearly we need $\chi(T) \leq 0$. But if $\chi(T) = 0$, then since $\pi(E^3/G - Y)$ is free and $\pi(T)$ is 2-generator not free, $q_*(T)$ is infinite cyclic, so ϕ cannot be onto (as G is 2-generator).

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Theorem Suppose that $b = c = d = 0$ (necessarily, $\chi(T) < 0$).
For OO, all realizable with S embedded.
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Some work in NN to show exceptions impossible. Need $\pi(E^3 - Y)$ is free and look carefully at presentations of $\pi(T)$.
For ON, the real point is that for odd $\chi(T)$, there will always be a orientation-reversing element z with $q_*(z) = 1$.

Only one kind of branching

Everything OK for $\chi(T) \leq 0$, but for $\chi(T) = 1, 2$ trouble if number of branch points at least 4 (necessarily even since). See example.

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Look at actions with reflections. Now orbifold has boundary.

But look out. Maybe "immersion" of S into E^3 is just a covering of S onto an embedded surface S' in E^3. Now you are asking whether, say, cyclic action on S' lifts to S.