Pentagonal geometries from maps on surfaces and their symmetries

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Joint work with Milagros Izquierdo and Marston Conder

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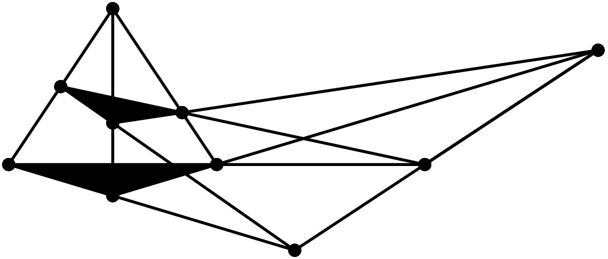
A plane configuration is a system of v points and b straight lines arranged in a plane in such a way that every point is incident with r lines and every line is incident with k points.
(Hilbert and Cohn-Vossen, Geometry and the Imagination,1932)



Theorem. (Desargues) In a projective plane, two triangles are perspective from a point if and only if they are perspective from a line.

The 10 points and 10 lines in Desargues' theorem form a configuration with high symmetry - any point could have been taken as the center of perspectivity.

Configurations may exist as combinatorial objects without there being any planar realization.

Let P be a set whose elements are called points.

Let L be a set of subsets of P whose elements are called lines.

Then (P, L) is called an **incidence geometry** or an **incidence structure**.

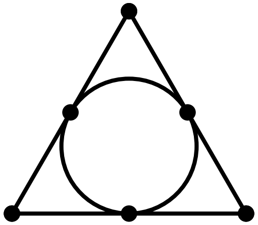
An incidence geometry(P, L) is a **partial linear space** if *at most one line passes through each pair of points.*

A partial linear space is

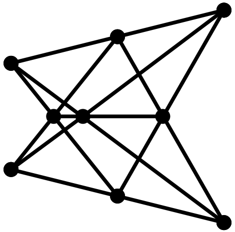
- r-**regular** if there are r lines through each point and
- k-**uniform** if there are k points on each line.

We say that it has **order** (r, k).

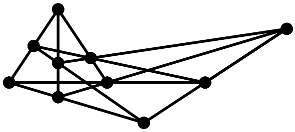
Regular and uniform partial linear spaces are also called **combinatorial configurations**.



A configuration of order $(r, k) = (2, 3)$.

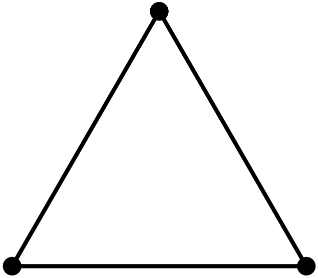


The Pappus' configuration has order $(r, k) = (3, 3)$.

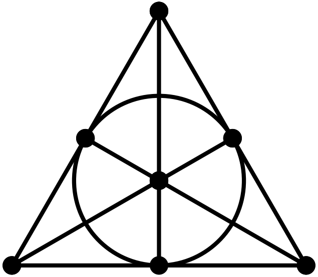


The Desargues' configuration has order $(r, k) = (3, 3)$.

Generalising the triangle: projective planes.



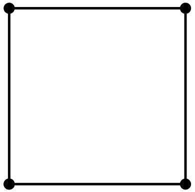
Triangle



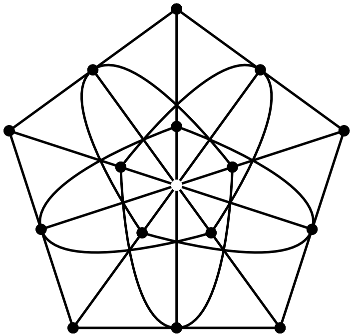
$PG(2,2)$ - of order 2

Geometric property: any two points are collinear and any two lines meet in one point.

Generalising the square: generalised quadrangles.



Square



$GQ(2,2)$
The Cremona-Richmond configuration

Geometric property: for any point x and line l with $x \notin l$ there is exactly one line m such that $x \in m$ and m and l meet.

Generalising the polygon: generalised polygons / Tits buildings of rank 2.

A generalised n-gon is a partial linear space such that its bipartite incidence (Levy) graph has

- diameter (longest distance between any vertices) $d = n$, and
- girth (length of shortest cycle) $g = 2n$.

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Equivalent (geometric) definition: A generalised n-gon is an incidence geometry such that

- there are no ordinary k-gons for $k < n$ as subgeometry,
- every pair of elements is contained in an ordinary n-gon,
- there exists an ordinary $(n+1)$-gon.

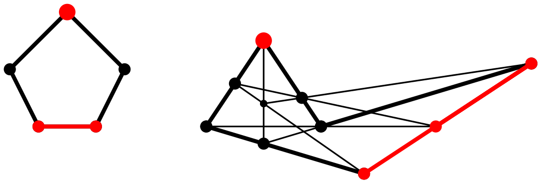
Theorem. (Feit-Higman, 1964) Generalised n-gons are either thin (i.e. with only two points per line or two lines per point) or $n = 3$, 4, 6 or 8.

So there are no generalised pentagons, nor generalised heptagons!

Is this the only way to generalise the polygon?

A pentagon is a partial linear space such that, for each point p, the points that are not collinear with p are on a unique line.

The Desargues' configuration also has this property:
fixing any center (point) of perspectivity p, the points not collinear with p are the points on the corresponding axis (line) of perspectivity.



Definition. (Ball-Bamberg-Devillers-S. 2013)
A pentagonal geometry is a regular and uniform partial linear space such that, for each point p, the points that are not collinear with x are on a unique line - the opposite line p^{opp}.

The deficiency graph of a partial linear space P is the graph G whose vertices are the points of P and two vertices x and y are connected iff x is **not** collinear to y in P.

Example.
The deficiency graph of the Desargues' configuration is the Petersen graph.

A **Moore graph** of girth 5 is a k-regular graph of girth 5 and diameter 2 (which is maximum).

Moore graphs are extremal graphs: given a diameter they have smallest possible girth.

Hoffman and Singleton proved that Moore graphs of odd girth g only exist if $g = 5$ and $k = 2, 3, 7$ and possibly 57.

Theorem. (Ball-Bamberg-Devillers-S. 2013) Pentagonal geometries of order (k, k) are all obtained from Moore graphs by tracing a (combinatorial) line through the neighborhood of each vertex. The Moore graph is the deficiency graph of the pentagonal geometry.

So the study of pentagonal geometries of order (k, k) is equivalent to the study of Moore graphs.

These pentagonal geometries are

- for $k = 2$: the ordinary pentagon gives: the ordinary pentagon,
- for $k = 3$: the Petersen graph gives: the Desargues' configuration,
- for $k = 7$: the Hoffman-Singleton graph gives: a pentagonal geometry of order $(7, 7)$.

Theorem. (Ball-Bamberg-Devillers-S. 2013) All pentagonal geometries of order $(k, k-1)$ are constructed from the pentagonal geometries of order (k, k) through the removal of one point and its opposite line.

Pentagonal geometries constructed through product constructions have **disconnected deficiency graph**.

The known pentagonal geometries with **connected deficiency graph** are of order

- $(r, 2)$,
- $(3, 3)$,
- $(7, 6)$,
- $(7, 7)$, and
- $(13, 3)$.

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A **map** is a drawing of a graph on a surface, such that the complement of the drawing is the disjoint union of finitely many topological discs called faces.

The **genus** of a map is the genus of the surface in which the graph is embedded and can be calculated using Eulers formula.

A map is

- **uniform** if there are p, q such that each face is a q-gon and p faces meet at each vertex, and
- **regular** if its automorphism group acts transitively on triples of incident vertices, edges and faces (flags). So regular implies uniform but the opposite is not true.

Gévay and Pisanski constructed spherical point-circle configurations coming from certain convex polytopes [Gévay-Pisanski, 2014].

Stereographic projection then gives point-circle configurations in the Euclidean plane.

They showed that all Platonic solids except the octahedron gives spherical point-circle configurations.

Only one of these is also a combinatorial point-line configuration: the $(20_3, 20_3)$-configuration constructed from the dodecahedron.

A uniform map on a surface is the quotient of a tiling of congruent polygons in a universal covering space U in congruent polygons by the action of a torsion-free subgroup of a triangle group $G \subseteq \Gamma(p, 2, q)$.

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Each circle contains p points and each point is on q circles, so this construction gives a configuration of points and circles on the surface!

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Theorem. (Izquierdo-S.) A uniform map on a surface yields a configuration of points and isometric circles on the same surface.

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The Petersen graph and the Desargues' configuration (order $(3, 3)$).

Desargues' configuration can be drawn *with crossings* (outside points of the configuration) in any projective plane over a field with enough elements.

It can be drawn *without crossings* on a non-orientable surface of Euler characteristic -5 [Coxeter, 1975], as a map of the collinearity graph (the Menger graph).

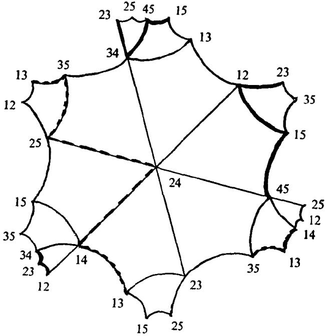


Figure from [Coxeter, 1975].

Every geodesic contains three edges forming a line of the configuration, so this is not barely a representation of Desargues' configuration, this IS Desargues' configuration. This map realizes the entire automorphism group of Desargues' configuration: The symmetric group on 5 elements S_5 (of order 120).

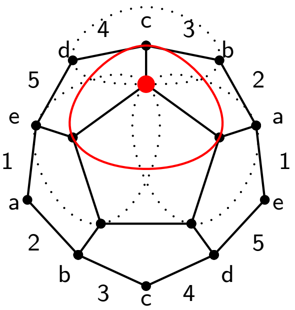
It is well-known that there is a map of the Petersen graph in the projective plane as the hemi-dodecahedron, obtained by identifying antipodal points in the dodecahedron on the sphere.

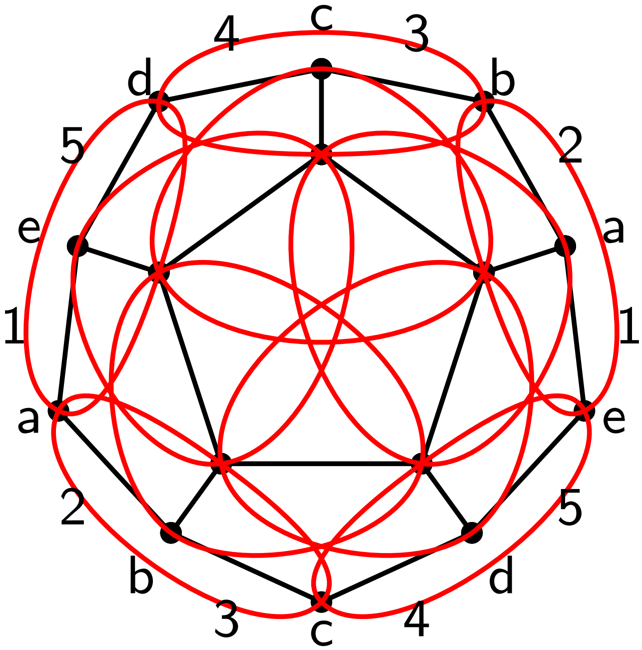
This is a “pentagonal embedding”: the polyhedron consists of 10 vertices, 15 edges and 6 pentagonal faces. It is a tessellation of the projective plane into 6 pentagonal faces.

For each vertex, the position of its three neighbors determine a circle.

Let these circles be the “lines” of a combinatorial configuration on the same point set as the graph.

Result: point-circle realization of Desargues' configuration in projective plane!

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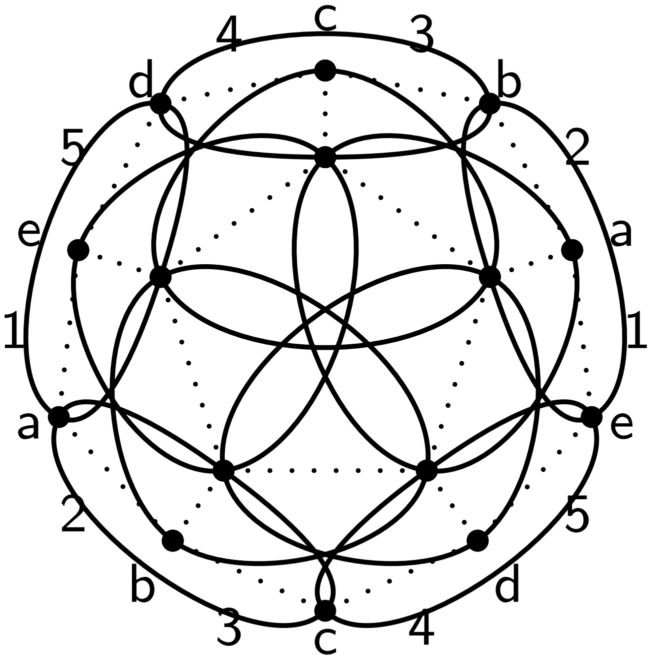
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Hoffman-Singleton graph and the pentagonal geometry of order $(7, 7)$.

Hoffman-Singleton graph:

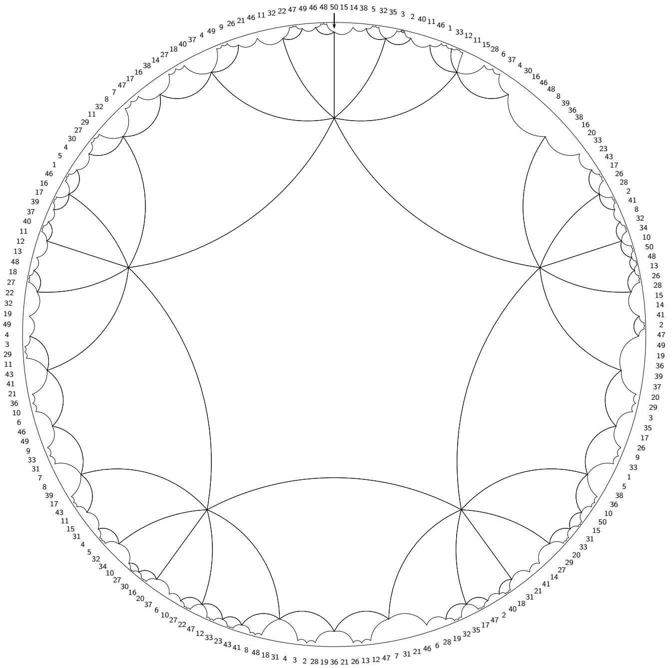
- Moore graph of girth 5,
- 50 vertices and 175 edges,
- automorphism group of graph is $PSU(3,5).2$ of order 252 000,
- largest possible characteristic (smallest genus) will be realized by a pentagonal embedding with 70 faces.
- Then Euler characteristic is $|V| - |E| + |F| = -55$: non-orientable.

Lemma. (Conder-S.) There is no regular embedding of the Hoffman-Singleton graph.

Theorem. (Conder-S.) There are uniform but non-regular pentagonal embedding of the Hoffman-Singleton graph in a surface of genus 57 with

- trivial automorphism group,
- 5-folded symmetry, and
- 7-folded symmetry.

Next slide: Pentagonal embedding of the Hoffman-Singleton graph in a surface of genus 57 with 5-folded symmetry, represented in the Poincaré disk. Numbers represent identifications of vertices. Vertex 50 is the vertex at the arrow and the rest are numbered in order.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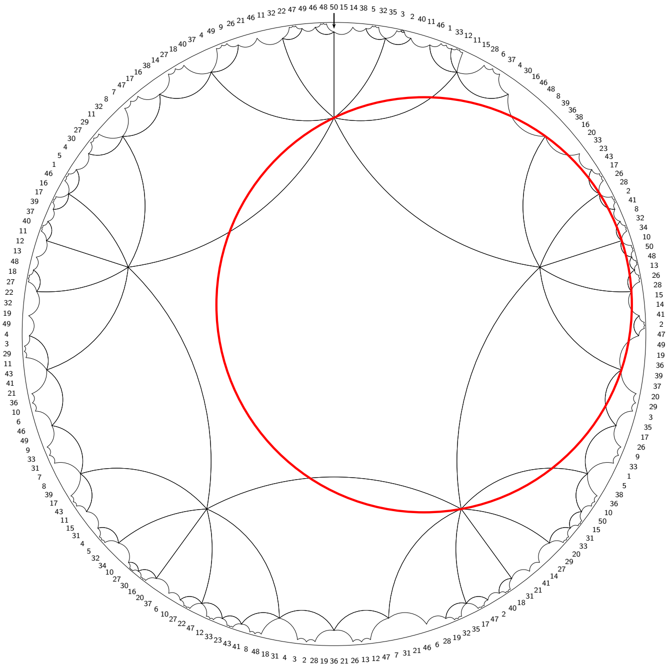
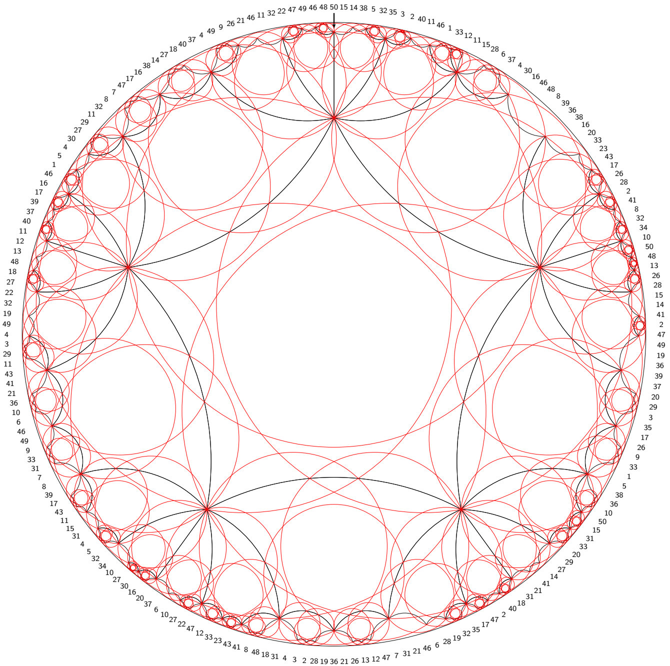
Now draw circles through the neighborhood of each vertex in the map.

Theorem. The pentagonal geometry of order $(7, 7)$ can be realized as a point-circle configuration on a surface of genus 57 with

- trivial automorphism group,
- 5-folded symmetry, or
- 7-folded symmetry.

Corollary. The pentagonal geometry of order $(7, 6)$ can also be realized as a point-circle configuration on a surface of genus 57.

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Thank you for listening!

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