Hereditary Polytopes

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Joint work with Mark Mixer and Asia Ivić Weiss.

Convex Regular Polytopes — Quick Review

Platonic solids $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, $\{5,3\}$

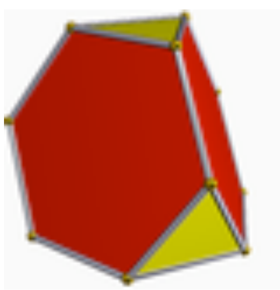
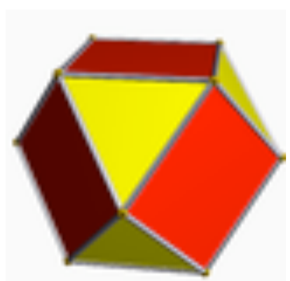
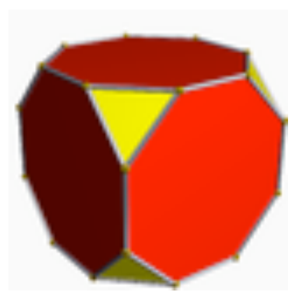
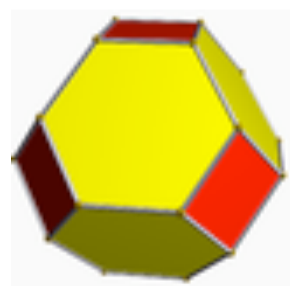
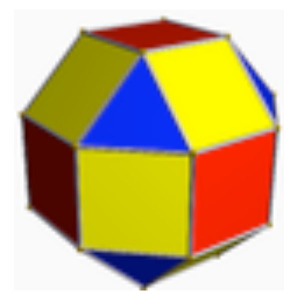
DIMENSION $n \geq 4$

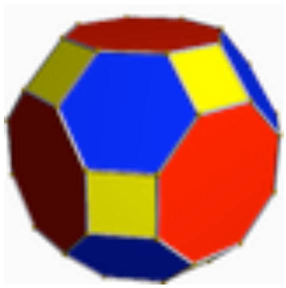
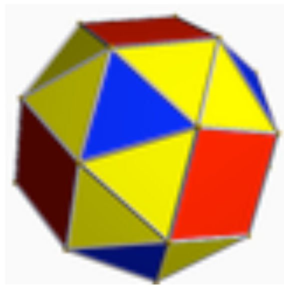
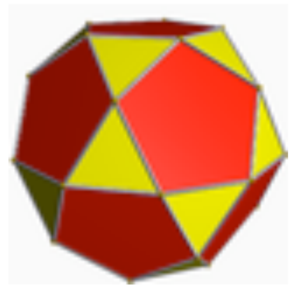
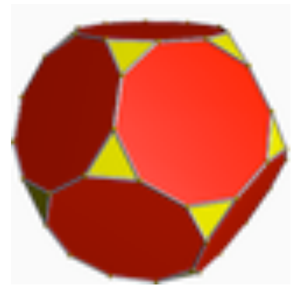
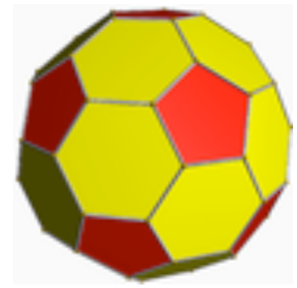
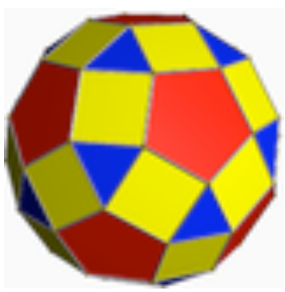
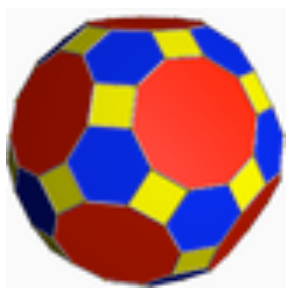
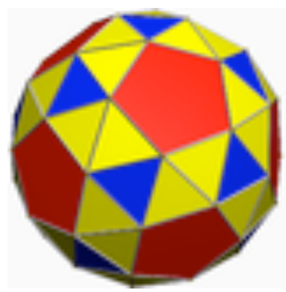
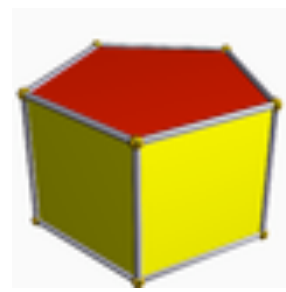
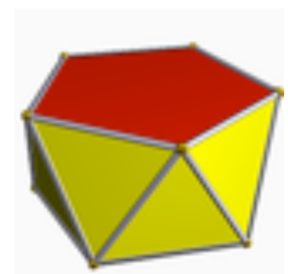
name	symbol	#facets	group	order
simplex	{3,3,3}	5	S_5	120
cross-polytope	{3,3,4}	16	B_4	384
cube	{4,3,3}	8	B_4	384
24-cell	{3,4,3}	24	F_4	1152
600-cell	{3,3,5}	600	H_4	14400
120-cell	{5,3,3}	120	H_4	14400
simplex	{3,. . . ,3}	n+1	S_{n+1}	$(n+1)!$
cross-polytope	{3,. . . ,3,4}	2^n	B_{n+1}	2^nn!
cube	{4,3,. . . ,3}	2n	B_{n+1}	2^nn!

Semiregular Convex Polytopes

Facets are *regular* (convex) polytopes. Geometric symmetry group is vertex-transitive.

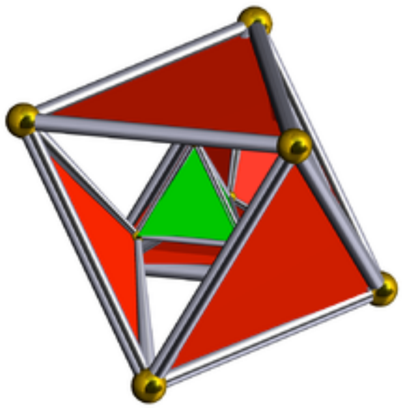
- Plane — regular polygons
- 3-space — Archimedean solids, prisms and antiprisms

- **Three polytopes for $n = 4$, and one each for $n = 5, 6, 7, 8$.**
- $n = 4$: $t_1\{3,3,3\}$, snub 24-cell, and $t_1\{3,3,5\}$.

  Schlegel diagram for $t_1\{3,3,3\}$

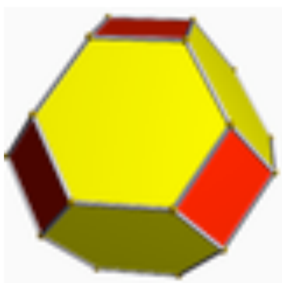
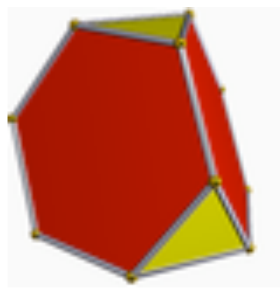


- $n = 5$: half-5-cube.
- Gosset polytopes 2_{21}, 3_{21}, 4_{21} related to the Coxeter groups E_6, E_7 and E_8.

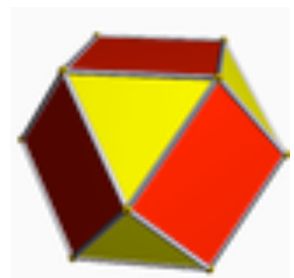
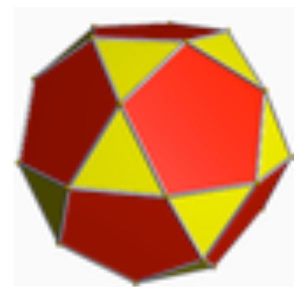
Vertices of 4_{21} are the 240 roots of E_8, with vertex-figure 3_{21}. Facets are 7-simplices and 7-crosspolytopes.

Semiregular polytopes are uniform polytopes.

Hereditary polytopes: *every symmetry of every facet is a symmetry of the polytope.*

No!

Yes!

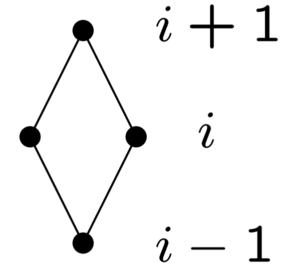
- Archimedean solids: only the cuboctahedron and icosidodecahedron are hereditary!

Abstract Polytopes P of rank n

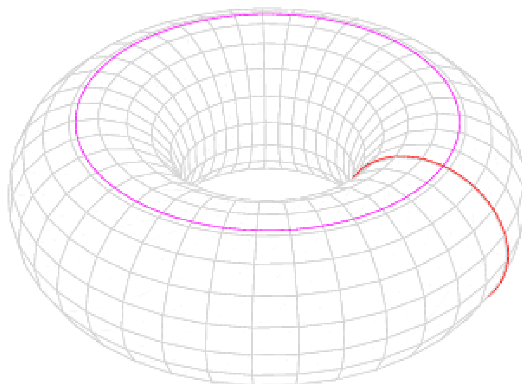
(Grünbaum, Danzer, 70's)

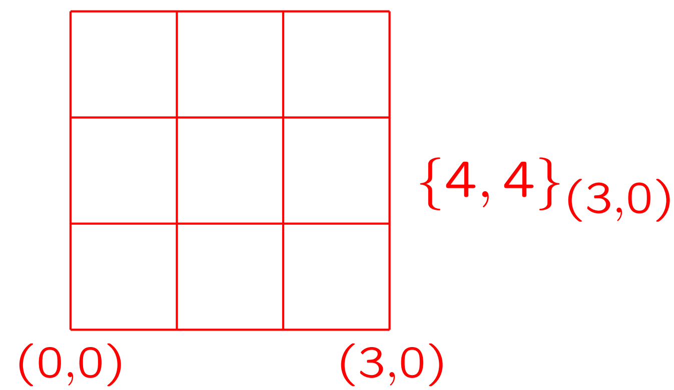
P	ranked partially ordered set
i-faces	elements of rank i (= -1,0,1,...,n)
i=0	vertices
i=1	edges
i=n-1	facets

- Faces F_{-1}, F_n (of ranks -1, n)
- Each flag of P contains exactly n+2 faces
- P is connected
- Intervals of rank 1 are diamonds:



Rank 3: Maps (2-cell tessellations) on closed surfaces.





P *regular*: Γ(P) flag transitive

P *chiral*: Γ(P) two flag-orbits, adjacent flags in different orbits

P *semi-regular*: P regular facets, Γ(P) vertex-transitive

P *hereditary*: *every* automorphism of *every* facet of P extends (uniquely) to an automorphism of P.

Polytopes with highly symmetric facets!

- Interesting case: regular facets, chiral facets, etc.

- Theorem: P hereditary, each facet of P regular or chiral. Then each facet regular or each facet chiral.

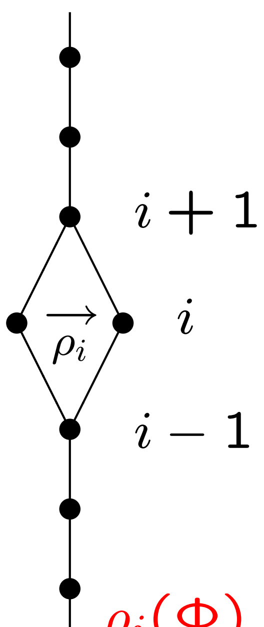
- No mixed type. P regular-facetted or P chiral-facetted.

- P hereditary, each facet $\{0,\ldots,i\}$-chain transitive for some $i \leq n-2$. Then P is $\{0,\ldots,i\}$-chain transitive (and hence its i-faces are regular and mutually isomorphic).

Hereditary n-polytopes with regular facets

- **Theorem:** P regular-facetted. Then P is hereditary iff P is regular or a two-orbit polytope of type $2_{\{0,\ldots,n-2\}}$.

Two-orbit polytopes of type 2_I, $I \subseteq \{0, \ldots, n-1\}$:



ρ_i exists precisely for $i \in I$

$\rho_i(\Phi) = \Phi^i$

Hubard (2008), Hubard, Orbanic & Weiss (2009)

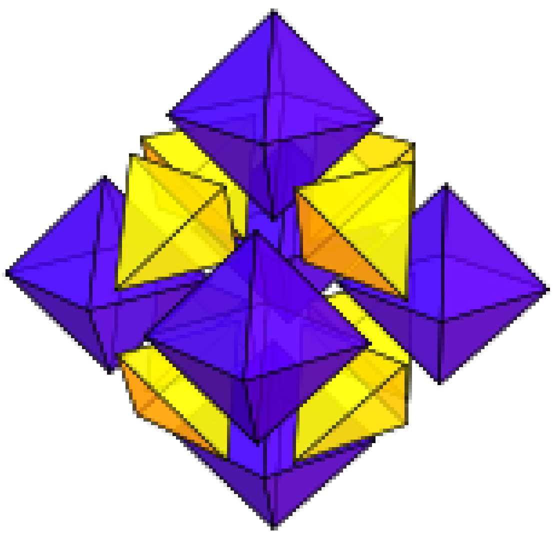
Rank 3 examples (regular, or 2-orbit of type $2_{\{0,1\}}$)

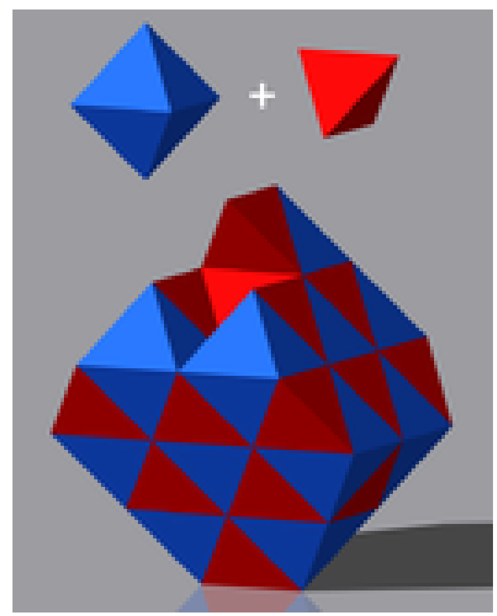
- medials of non-selfdual regular polyhedra $\{p, q\}$
  - p-gonal and q-gonal faces, 4-valent vertices
  - cube $\longrightarrow$ cuboctahedron

- derived from bipartite regular maps K of type $\{2r, q\}$
  - r-gonal faces inscribed in old $2r$-gonal faces, and q-gonal faces as old vertex-figures
  - $2q$-valent vertices
  - notation: K^a ("a" for "alternating")

Rank 4 examples (regular, or 2-orbit of type $2_{\{0,1,2\}}$)

- semi-regular tessellation of $\mathbb{E}^3$ by regular tetrahedra and regular octahedra (also 4-toroids related to this)





Hereditary n-polytopes with chiral facets

• Theorem: A chiral-facetted hereditary n-polytope either is a 2-orbit polytope which is itself chiral or is of type $2_{\{n-1\}}$, or is a 4-orbit polytope.

Representative flags under $\Gamma(P)$ from among

$$\Phi, \Phi^{j}, \Phi^{n-1}, \Phi^{n-1,j} \quad (j = 0, \ldots, n-2).$$

Chiral-facetted, chiral polytopes: many examples

S. & Weiss (rank 4, early 1990's), Monson & Weiss (rank 4, 1990's), Conder, Hubard & Pisanki (rank 5, 2008), Pellicer (any rank, 2009)

Every chiral polytope occurs as a facet of a chiral polytope.
(Cunningham & Pellicer 2013, S. & Weiss 1994)

Chiral-facetted hereditary 2-orbit polytopes of type $2_{\{n-1\}}$

- Only ρ_{n-1} is in $\Gamma(P)$! Isomorphic facets! Equifacetted!

- Examples from the power-polytope construction 2^K (Danzer).

K a chiral $(n-1)$-polytope, K facet-describable, K^* its dual.

The dual of 2^{K^*} is hereditary, chiral-facetted with facets isomorphic to K, 2-orbit of type $2_{\{n-1\}}$, with group

$$C_2 \wr \Gamma(K) = C_2^f \rtimes \Gamma(K), \quad f := \#\text{facets of } K.$$

Example: $K = \{4,4\}_{(1,3)}$
4-polytope $(2^{K^*})^*$ of type Schläfli type $\{4,4,4\}$, with facets $\{4,4\}_{(1,3)}$, vertex-figures $\{4,4\}_{(4,0)}$, and group $C_2^{10} \rtimes [4,4]_{(1,3)}$.

Power Polytopes 2^K

- 2^K regular if K regular

- 2^K is k-orbit if K is k-orbit ($k \geq 1$)

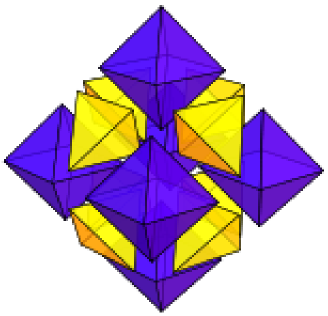
- 2^K is 2-orbit of type $J := \{0\} \cup \{i+1 | i \in I\}$ if K is 2-orbit of type I.

- 2^K is 2-orbit of type $\{0\}$ if K is chiral

Chiral-facetted hereditary 4-orbit polytopes

Blueprint

• Relationship between two tilings in $\mathbb{E}^3$: the cubical tessellation C (bipartite!), and the semiregular tessellation T by regular octahedra and regular tetrahedra.



• Color vertices of C red and yellow! Vertices of T are the yellow vertices of C.

• Two kinds of tiles (facets) in T: octahedral vertex-figures of C at red vertices, and tetrahedra inscribed in tiles of C.

Note: T regular-facetted, 2-orbit of type $2_{\{0,1,2\}}$, not 4-orbit.

Old polytope P

- P regular or chiral 4-polytope of type $\{4, q, r\}$, with facets K and vertex-figures L

- edge graph of P bipartite (yellow and red vertices)

- P, L vertex-describable, and opposite vertices in a 2-face are never opposite vertices of another 2-face

New polytope P^a

- two kinds of facets:

– vertex-figures of P at the red vertices, isomorphic to L
– polyhedra F^a ($\cong \mathcal{K}^a$), with F a (bipartite!) facet of P

- vertex-figures isomorphic to medials Me(L)

- P^a hereditary

- P^a alternating, four facets around an edge

- P^a regular-facetted if K and L are regular
(P^a semiregular, alternating — Monson & S.)

- P^a chiral-facetted if K and L are chiral

- P^a chiral-facetted and 4-orbit if K, L chiral and $L \not\cong K^a$
(for example, if $|\Gamma(L)| \neq |\Gamma(K)|/2$)

- $\Gamma^c(P) \leq \Gamma(P^a)$, index 1 or 2, same or twice the number of flag-orbits as $\Gamma(P^a)$

Example: $\mathsf{P} = \{\{4,4\}_{(1,3)}, \{4,4\}_{(1,3)}\}$ (bipartite!)

P^a has facets $\{4,4\}_{(1,3)}$ and $\{4,4\}_{(1,2)} = \{4,4\}^a_{(1,3)}$, and vertex-figures $\{4,4\}_{(2,4)} = \mathrm{Me}(\{4,4\}_{(1,3)})$.

Work in progress (with Antonio Montero, Luis Ruiz, Asia Ivić Weiss)

- *Chirally hereditary polytopes:* every *rotational* automorphism of every facet is a global automorphism.

Chiral polytopes are chirally hereditary, even those with regular facets.

- *(Strongly) j-face hereditary polytopes ($1 \leq j \leq n-1$):* every automorphism of every j-face extends to a global automorphism (fixing every face containing the j-face, resp.).

- *Inductively hereditary polytopes:* each face of rank at least 3 (including P itself) is a non-regular hereditary polytope.

• *Geometrically hereditary polytopes:* *convex* polytopes in $\mathbb{E}^k$ that inherit all the geometric symmetries of each of its facets.

Completely unexplored!

Extension problems: To which extent can one preassign hereditary polytopes as facets of hereditary polytopes of higher rank?

- Can show: Every finite vertex-describable j-face hereditary n-polytope P is the vertex-figure of vertex-transitive $(j+1)$-face hereditary $(n+1)$-polytope. (Take 2^P !)

.... The End ....

Thank you!

Abstract

Every regular polytope has the remarkable property that it inherits all symmetries of each of its facets. This property distinguishes a natural class of polytopes which are called hereditary. In this talk we present the basic theory of hereditary polytopes, focussing on the analysis and construction of hereditary polytopes with highly symmetric faces.