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Self-Complementary Metacirculants

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(joint work with Cai Heng Li and Shu Jiao Song)

The University of Western Australia

July 7, 2014

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- A *circulant* is a Cayley graph of a cyclic group.
- A group R is *metacyclic* if exists $N \lhd R$ such that $N, R/N$ are cyclic. Γ is a *metacirculant* if $R <$ AutΓ is transitive and metacyclic.

What is a self-complementary vertex-transitive graph?

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Self-complementary vertex-transitive (SCVT) graphs are the graphs that are both self-complementary and vertex-transitive.

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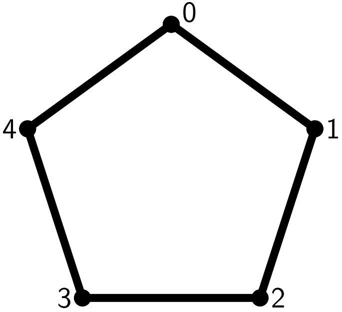

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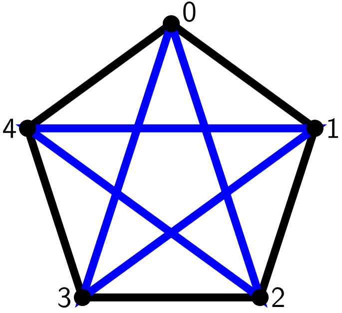

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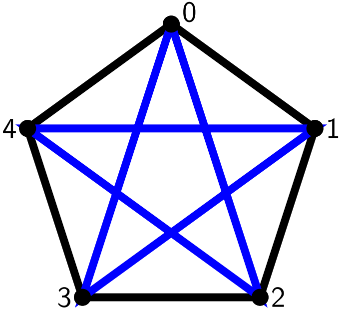

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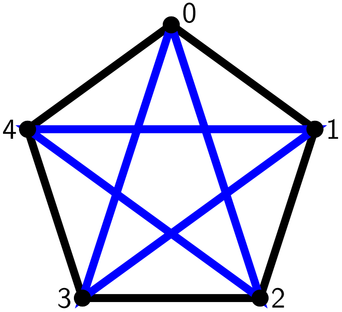

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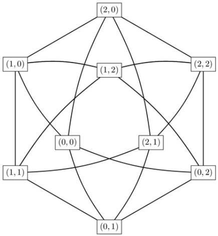
An isomorphism $\sigma : \Gamma \to \overline{\Gamma}$ is a *complementing isomorphism*.

- $o(\sigma) = 2^e$;
- σ does not fix any edge $\Rightarrow$ $4 \mid o(\sigma)$;
- $\sigma^2 \in \mathrm{Aut}\Gamma \;\Rightarrow\; \sigma$ normalises $\mathrm{Aut}\Gamma$.

- Observation



$\Gamma = \mathrm{Cay}(R, S)$ is an SCVT-graph.

$$R = \mathbb{Z}_3^2, \quad S = \{(0,1), (2,0), (0,2), (1,0)\}$$

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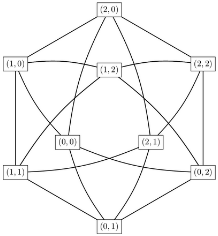

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Define

$$\sigma = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \in \mathrm{GL}(2,3).$$

Then $S^\sigma = R^\# \setminus S$ and $S \bigcup S^\sigma = R^\#$. □

Background

- (Sachs, 1962) SC-graphs
  - properties of SC-graphs
  - construction method for SC-graphs
  - conditions for the order of SC-circulants

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  - adjacency matrices
  - small order graphs

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- (Li and Rao, 2014) SCVT-graphs of order pq.

Why study SCVT-graphs?

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- The *diagonal Ramsey number* ‣ Example

$$R(n,n) = \min\{|V\Gamma| : \mathrm{K}_n \leq \Gamma \text{ or } \mathrm{K}_n \leq \overline{\Gamma}\}$$

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$$\Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k\to\infty} \sqrt[k]{\alpha(\Gamma^k)}$$

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- (Beezer, 2006) In 50,502,031,367,952 non-isomorphic graphs of order 13, only 2 of them are SCVT.

Construction of Self-Complementary Cayley Graphs

Observation

Let R be a group and $\sigma \in \mathrm{Aut}(R)$. Then

$$\mathrm{Cay}(R, S) \overset{\sigma}{\cong} \mathrm{Cay}(R, S^{\sigma}).$$

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$$\mathrm{Cay}(R, S) \overset{\sigma}{\cong} \mathrm{Cay}(R, S^\sigma).$$

If there exists $\sigma \in \mathrm{Aut}(R)$ such that

$$S^\sigma = R^\# \setminus S,$$

then

$$\Gamma = \mathrm{Cay}(R, S) \cong \mathrm{Cay}(R, S^\sigma) \cong \mathrm{Cay}(R, R^\# \setminus S) = \overline{\Gamma}.$$

▸ Example

Properties of σ

(i) σ does not fix any element of $R^{\#}$ $\Rightarrow$ σ is *fixed-point-free*.

(ii) σ^2 fixes S and $R^{\#} \setminus S$ $\Rightarrow$ $\sigma^2 \in \mathrm{Aut}\mathit{\Gamma}$ $\Rightarrow$ σ normalises $\mathrm{Aut}\mathit{\Gamma}$.

(iii) $o(\sigma) = 2^e, e \geq 2$.

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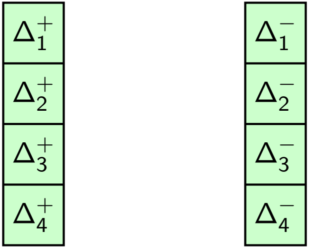
Construction 1

1. $\langle \sigma^2 \rangle$*-orbits on* $R^{\#}$*:* $\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ *where* $(\Delta_i^+)^\sigma = \Delta_i^-$ *for each* i*;*



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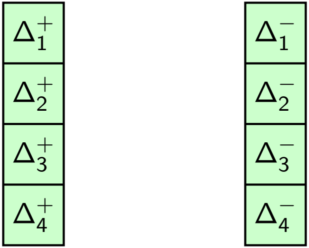

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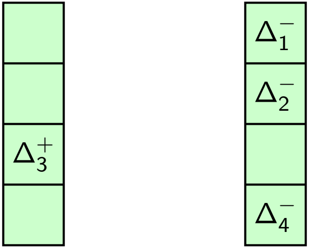

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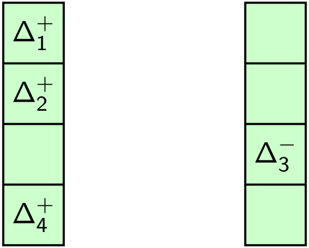

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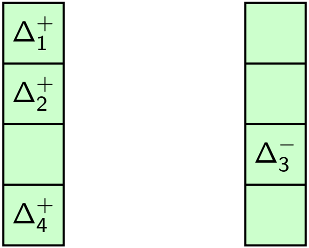

Then $\mathrm{Cay}(R, S)$ is self-complementary.

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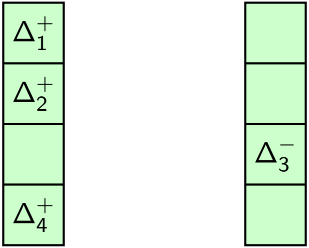

Construction 2

Let $p \equiv 1$ *or* $9 \pmod{40}$, $R = \mathbb{Z}_p^2$, *and let* $\sigma \in \mathbf{Z}(\mathrm{GL}(2,p))$ *with* $8 \mid o(\sigma)$. *Let* $H = \langle \sigma, \mathrm{SL}(2,5)\rangle$, $M = \langle \sigma^2, \mathrm{SL}(2,5)\rangle$.

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$\Gamma = \mathrm{Cay}(R, S)$ is an SC-metacirculant with $\mathrm{Aut}\,\Gamma$ insoluble.

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Lemma 3

There exist SC-metacirculants of order $p_1^2 \ldots p_t^2$ (p_i distinct) with $\mathrm{Aut}\Gamma \geq \mathbb{Z}_{p_1 \ldots p_t}^2{:}(\mathbb{Z}_\ell \circ \mathrm{SL}(2,5)^t)$.

Self-complementary metacirculants

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The automorphism group of a self-complementary circulant is soluble.

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The automorphism group of a self-complementary circulant is soluble.

The following theorem extends the result.

Theorem 5 (Li, Rao and Song, 2014)

The automorphism group of a self-complementary metacirculant is either soluble, or contains composition factor A_5*.*

Proof of theorem 5

- Γ is a self-complementary metacirculant
- σ is a complementing isomorphism
- $G = \mathrm{Aut}\Gamma$
- $X = \langle G, \sigma \rangle = G.\mathbb{Z}_2$
- $R < G$ is transitive and metacyclic

Proof of theorem 5

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A block system $\mathcal{B}$ is a nontrivial X-invariant partition of V.

(i) V has no block systems $\Rightarrow$ X is primitive.

(ii) V has a block system $\Rightarrow$ X is imprimitive.

The primitive case

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Theorem 6 (Guralnick, Li, Praeger and Saxl, 2004)

If X is primitive, then

(i) *X is affine, or*

(ii) *X is of product action type with* $\mathrm{soc}(X) = \mathrm{PSL}(2, q^2)^\ell$.

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1. R is metacyclic $\Rightarrow$ X is affine of dimension ≤ 2.
2. X is insoluble $\Rightarrow$ $X = \mathbb{Z}_p^2{:}(\mathbb{Z}_\ell \circ \mathrm{SL}(2, 5))$ construction

The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If X is imprimitive, then:

(i) $[B]_\Gamma$ *is self-complementary,* $G_B^B \leq \mathrm{Aut}[B]_\Gamma$*, and* σ^B *is its complementing isomorphism;*

(ii) *there is a self-complementary graph* Σ *with vertex set* $\mathcal{B}$ *such that* $G^{\mathcal{B}} \leq \mathrm{Aut}\Sigma$ *and each element of* $X^{\mathcal{B}} \setminus G^{\mathcal{B}}$ *is its complementing isomorphism.*

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(i) $[B]_\Gamma$ *is self-complementary,* $G_B^B \leq \mathrm{Aut}[B]_\Gamma$, *and* σ^B *is its complementing isomorphism;*

(ii) *there is a self-complementary graph* Σ *with vertex set* $\mathcal{B}$ *such that* $G^{\mathcal{B}} \leq \mathrm{Aut}\Sigma$ *and each element of* $X^{\mathcal{B}} \setminus G^{\mathcal{B}}$ *is its complementing isomorphism.*

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4. Consider $K^B \leq X_B^B$ and $K \leq K^{B_1} \times \ldots \times K^{B_2}$.
5. $R^{\mathcal{B}} \leq X^{\mathcal{B}}$ is transitive and metacyclic. □

Conjecture 8

Self-complementary metacirculants are Cayley graphs? ▸ Example

Thank you!