Self-Complementary Metacirculants

Grant Rao
(joint work with Cai Heng Li and Shu Jiao Song)

The University of Western Australia

July 7, 2014
All graphs are assumed to be finite simple.

- $\Gamma$ is *vertex-transitive* if $\text{Aut}\Gamma$ is transitive on $V$. 

- A *circulant* is a Cayley graph of a cyclic group.

- A group $R$ is *metacyclic* if there exists $N \triangleleft R$ such that $N$, $R/N$ are cyclic.

- $\Gamma$ is a *metacirculant* if $R < \text{Aut}\Gamma$ is transitive and metacyclic.
All graphs are assumed to be finite simple.

- $\Gamma$ is vertex-transitive if $\text{Aut}\Gamma$ is transitive on $V$.
- For a finite group $R$, let $R^\# := R \setminus \{1\}$, $S \subseteq R^\#$. 

All graphs are assumed to be finite simple.

- $\Gamma$ is **vertex-transitive** if $\text{Aut}\Gamma$ is transitive on $V$.
- For a finite group $R$, let $R^\# := R \setminus \{1\}$, $S \subseteq R^\#$. $\Gamma = \text{Cay}(R, S)$ is a **Cayley graph** where

  \[ V = R, \]
  \[ E = \{(a, b) | ba^{-1} \in S\} \]
All graphs are assumed to be finite simple.

- $\Gamma$ is \textit{vertex-transitive} if $\text{Aut}\Gamma$ is transitive on $V$.
- For a finite group $R$, let $R^\# := R \setminus \{1\}$, $S \subseteq R^\#$. $\Gamma = \text{Cay}(R, S)$ is a \textit{Cayley graph} where

  $$
  V = R, \quad E = \{(a, b) | ba^{-1} \in S\}
  $$

  $\Gamma$ undirected $\iff$ $S = S^{-1} = \{s^{-1} | s \in S\}$.\]
All graphs are assumed to be finite simple.

- Γ is **vertex-transitive** if AutΓ is transitive on V.
- For a finite group R, let $R^\# := R \setminus \{1\}$, $S \subseteq R^\#$. $\Gamma = \text{Cay}(R, S)$ is a **Cayley graph** where

  \[
  V = R, \\
  E = \{(a, b)|ba^{-1} \in S}\]

  $\Gamma$ undirected $\Leftrightarrow S = S^{-1} = \{s^{-1}|s \in S\}$.

- A **circulant** is a Cayley graph of a cyclic group.
All graphs are assumed to be finite simple.

- $\Gamma$ is **vertex-transitive** if $\text{Aut}\Gamma$ is transitive on $V$.
- For a finite group $R$, let $R^\# := R \setminus \{1\}$, $S \subseteq R^\#$. 
  $\Gamma = \text{Cay}(R, S)$ is a **Cayley graph** where

  $$V = R,$$

  $$E = \{(a, b)|ba^{-1} \in S\}$$

  $\Gamma$ undirected $\iff S = S^{-1} = \{s^{-1}|s \in S\}$.

- A **circulant** is a Cayley graph of a cyclic group.

- A group $R$ is **metacyclic** if exists $N \triangleleft R$ such that $N, R/N$ are cyclic. $\Gamma$ is a **metacirculant** if $R \lhd \text{Aut}\Gamma$ is transitive and metacyclic.
What is a self-complementary vertex-transitive graph?

\( \Gamma \) is called \textit{self-complementary (SC)} if \( \Gamma \cong \overline{\Gamma} \).
What is a self-complementary vertex-transitive graph?

$\Gamma$ is called self-complementary (SC) if $\Gamma \cong \overline{\Gamma}$.

Self-complementary vertex-transitive (SCVT) graphs are the graphs that are both self-complementary and vertex-transitive.
What is a self-complementary vertex-transitive graph?

Γ is called \textit{self-complementary (SC)} if \( \Gamma \cong \overline{\Gamma} \).
Γ is called **self-complementary (SC)** if $\Gamma \cong \overline{\Gamma}$.
What is a self-complementary vertex-transitive graph?

\( \Gamma \) is called \textit{self-complementary (SC)} if \( \Gamma \cong \bar{\Gamma} \).

\[ \sigma = (1342) \text{ maps } \Gamma \text{ to } \bar{\Gamma} \]
What is a self-complementary vertex-transitive graph?

\( \Gamma \) is called **self-complementary (SC)** if \( \Gamma \cong \overline{\Gamma} \).

\[ \sigma = (1342) \text{ maps } \Gamma \text{ to } \overline{\Gamma} \]

An isomorphism \( \sigma : \Gamma \to \overline{\Gamma} \) is a **complementing isomorphism**.
What is a self-complementary vertex-transitive graph?

An isomorphism $\sigma : \Gamma \to \bar{\Gamma}$ is a *complementing isomorphism*.

- $o(\sigma) = 2^e$;
- $\sigma$ does not fix any edge $\Rightarrow$ $4 \mid o(\sigma)$;
- $\sigma^2 \in \text{Aut}\Gamma \Rightarrow \sigma$ normalises $\text{Aut}\Gamma$. 
Observation

$\Gamma = \text{Cay}(R, S)$ is an SCVT-graph.

$R = \mathbb{Z}_3^2, \quad S = \{(0, 1), (2, 0), (0, 2), (1, 0)\}$
\( \Gamma = \text{Cay}(R, S) \) is an SCVT-graph.

\[ R = \mathbb{Z}_3^2, \quad S = \{(0, 1), (2, 0), (0, 2), (1, 0)\} \]

Define

\[ \sigma = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \in \text{GL}(2, 3). \]

Then \( S^\sigma = R^\# \setminus S \) and \( S \cup S^\sigma = R^\# \). \( \square \)
Background

- (Sachs, 1962) SC-graphs
  - properties of SC-graphs
  - construction method for SC-graphs
  - conditions for the order of SC-circulants

- (Zelinka, 1979) The order of an SCVT-graph $\equiv 1 \pmod{4}$.

- (Rao, 1985) Regular and strongly regular SC-graphs.

- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.

- (Mathon, 1988) Strongly regular SC-graphs


- (Muzychuk, 1999) An SCVT-graph of order $n$ exists $\iff n^p \equiv 1 \pmod{4}$ for all $p$.

- (Li, Sun and Xu, 2014) SC-circulants of prime-power order.

- (Li and Rao, 2014) SCVT-graphs of order $pq$. 
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph $\equiv 1 \pmod{4}$.
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph $\equiv 1 \pmod{4}$.
- (Rao, 1985) Regular and strongly regular SC-graphs.
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph \( \equiv 1 \pmod{4} \).
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph \( \equiv 1 \pmod{4} \).
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
- (Mathon, 1988) Strongly regular SC-graphs
  - adjacency matrices
  - small order graphs
- (Muzychuk, 1999) An SCVT-graph of order \( n \) exists if and only if \( n p \equiv 1 \pmod{4} \) for all primes \( p \).
- (Li, Sun and Xu, 2014) SC-circulants of prime-power order.
- (Li and Rao, 2014) SCVT-graphs of order \( pq \).
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph $\equiv 1 \pmod{4}$.
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
- (Mathon, 1988) Strongly regular SC-graphs
- (Fronček, Rosa and Širáň, 1996)
  (Alspach, Morris and Vilfred, 1999)
  The orders of SC-circulants.
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph \( \equiv 1 \pmod{4} \).
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
- (Mathon, 1988) Strongly regular SC-graphs
- (Fronček, Rosa and Širáň, 1996)
  (Alspach, Morris and Vilfred, 1999)
  The orders of SC-circulants.
- (Muzychuk, 1999) An SCVT-graph of order \( n \) exists
  \( \iff n_p \equiv 1 \pmod{4} \ \forall p \).
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph \( \equiv 1 \pmod{4} \).
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
- (Mathon, 1988) Strongly regular SC-graphs
- (Muzychuk, 1999) An SCVT-graph of order \( n \) exists \( \iff n_p \equiv 1 \pmod{4} \ \forall p \).
- (Li, Sun and Xu, 2014) SC-circulants of prime-power order.
Background

- (Sachs, 1962) SC-graphs
- (Zelinka, 1979) The order of an SCVT-graph $\equiv 1 \pmod{4}$.
- (Rao, 1985) Regular and strongly regular SC-graphs.
- (Suprunenko, 1985) Construct SC-graphs using group theoretical method.
- (Mathon, 1988) Strongly regular SC-graphs
- (Fronček, Rosa and Širáň, 1996)
  (Alspach, Morris and Vilfred, 1999)
  The orders of SC-circulants.
- (Muzychuk, 1999) An SCVT-graph of order $n$ exists
  $\iff n_p \equiv 1 \pmod{4} \ \forall p$.
- (Li, Sun and Xu, 2014) SC-circulants of prime-power order.
- (Li and Rao, 2014) SCVT-graphs of order $pq$. 
Why study SCVT-graphs?

- The *diagonal Ramsey number*

\[ R(n, n) = \min \{|V\Gamma| : K_n \leq \Gamma \text{ or } K_n \leq \Gamma'\} \]

- Example

- Homogeneous factorisations, transitive orbital decompositions


- (Beezer, 2006) In 50,502,031,367,952 non-isomorphic graphs of order 13, only 2 of them are SCVT.
Why study SCVT-graphs?

- The **diagonal Ramsey number**

\[ R(n, n) = \min\{|V\Gamma| : K_n \leq \Gamma \text{ or } K_n \leq \bar{\Gamma}\} \]

- The **Shannon capacity**

\[ \Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(\Gamma^k)} \]
Why study SCVT-graphs?

- The **diagonal Ramsey number**

\[ R(n, n) = \min\{ |V(\Gamma)| : K_n \leq \Gamma \text{ or } K_n \leq \bar{\Gamma} \} \]

- The **Shannon capacity**

\[ \Theta(\Gamma) = \sup_k k \sqrt{\alpha(\Gamma^k)} = \lim_{k \to \infty} k \sqrt{\alpha(\Gamma^k)} \]

(Lovász, 1979) \( \Theta(\text{an SCVT-graph of order } n) = \sqrt{n} \)
Why study SCVT-graphs?

- The *diagonal Ramsey number*

\[ R(n, n) = \min\{|V\Gamma| : K_n \leq \Gamma \text{ or } K_n \leq \bar{\Gamma}\} \]

- The *Shannon capacity*

\[ \Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(\Gamma^k)} \]

(Lovász, 1979) \( \Theta(\text{an SCVT-graph of order } n) = \sqrt{n} \).

- Homogeneous factorisations, *transitive orbital decompositions*
Why study SCVT-graphs?

▶ The *diagonal Ramsey number*  

$$R(n, n) = \min\{|V\Gamma| : K_n \leq \Gamma \text{ or } K_n \leq \Gamma\}$$

▶ The *Shannon capacity*

$$\Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(\Gamma^k)}$$

(Lovász, 1979) $\Theta$(an SCVT-graph of order $n) = \sqrt{n}$.

▶ *Homogeneous factorisations, transitive orbital decompositions*

▶ (Zhang, 1992) Algebraic characterisation of arc-transitive SC-graphs.


▶ (Beezer, 2006) In 50,502,031,367,952 non-isomorphic graphs of order 13, only 2 of them are SCVT.
Why study SCVT-graphs?

- The *diagonal Ramsey number*

\[ R(n, n) = \min\{|V\Gamma| : K_n \leq \Gamma \text{ or } K_n \leq \overline{\Gamma}\} \]

- The *Shannon capacity*

\[ \Theta(\Gamma) = \sup_k \sqrt[k]{\alpha(\Gamma^k)} = \lim_{k \to \infty} \sqrt[k]{\alpha(\Gamma^k)} \]

(Lovász, 1979) \( \Theta(\text{an SCVT-graph of order } n) = \sqrt{n} \).

- *Homogeneous factorisations, transitive orbital decompositions*


- (Beezer, 2006) In 50,502,031,367,952 non-isomorphic graphs of order 13, only 2 of them are SCVT.
Construction of Self-Complementary Cayley Graphs

Observation
Let $R$ be a group and $\sigma \in \text{Aut}(R)$. Then

$$\text{Cay}(R, S) \overset{\sigma}{\cong} \text{Cay}(R, S^\sigma).$$
Construction of Self-Complementary Cayley Graphs

Observation
Let $R$ be a group and $\sigma \in \text{Aut}(R)$. Then

$$\text{Cay}(R, S) \cong \text{Cay}(R, S^\sigma).$$

If there exists $\sigma \in \text{Aut}(R)$ such that

$$S^\sigma = R^\# \setminus S,$$

then

$$\Gamma = \text{Cay}(R, S) \cong \text{Cay}(R, S^\sigma) \cong \text{Cay}(R, R^\# \setminus S) = \overline{\Gamma}.$$
Properties of $\sigma$

(i) $\sigma$ does not fix any element of $R^\#$ $\Rightarrow$ $\sigma$ is fixed-point-free.

(ii) $\sigma^2$ fixes $S$ and $R^\# \setminus S$ $\Rightarrow$ $\sigma^2 \in \text{Aut} \Gamma$ $\Rightarrow$ $\sigma$ normalises $\text{Aut} \Gamma$.

(iii) $\phi(\sigma) = 2^e$, $e \geq 2$. 
Construction 1

1. $\langle \sigma^2 \rangle$-orbits on $R^\#: \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ where $(\Delta_i^+)^\sigma = \Delta_i^-$ for each $i$;
Construction 1

1. $\langle \sigma^2 \rangle$-orbits on $R^\#: \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ where $(\Delta_i^+)^\sigma = \Delta_i^-$ for each $i$;
2. $S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}$ where $\varepsilon_i = +$ or $-$.  

<table>
<thead>
<tr>
<th>$\Delta_1^+$</th>
<th>$\Delta_1^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_2^+$</td>
<td>$\Delta_2^-$</td>
</tr>
<tr>
<td>$\Delta_3^+$</td>
<td>$\Delta_3^-$</td>
</tr>
<tr>
<td>$\Delta_4^+$</td>
<td>$\Delta_4^-$</td>
</tr>
</tbody>
</table>
Construction 1

1. \( \langle \sigma^2 \rangle \)-orbits on \( R^\# \): \( \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^- \) where 
   \( (\Delta_i^+)^\sigma = \Delta_i^- \) for each \( i \);

2. \( S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i} \) where \( \varepsilon_i = + \) or \( - \).
Construction 1

1. \( \langle \sigma^2 \rangle \)-orbits on \( R^\# \): \( \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^- \) where \( (\Delta_i^+)\sigma = \Delta_i^- \) for each \( i \);

2. \( S = \bigcup_{i=1}^r \Delta_{i}^{\epsilon_i} \) where \( \epsilon_i = + \) or \(-\).
Construction 1

1. $\langle \sigma^2 \rangle$-orbits on $R^\#$: $\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ where $(\Delta_i^+)^\sigma = \Delta_i^-$ for each $i$;

2. $S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}$ where $\varepsilon_i = +$ or $-$. 

Then $\text{Cay}(R, S)$ is self-complementary.
Construction 1

1. \(\langle \sigma^2 \rangle\)-orbits on \(R^\#\): \(\Delta^+_1, \Delta^-_1, \Delta^+_2, \Delta^-_2, \ldots, \Delta^+_r, \Delta^-_r\) where \((\Delta_i^+)\sigma = \Delta^-_i\) for each \(i\);

2. \(S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}\) where \(\varepsilon_i = +\) or \(-\).

\[
\begin{array}{c}
\Delta^+_1 \\
\Delta^+_2 \\
\Delta^+_4 \\
\end{array}
\quad
\begin{array}{c}
\Delta^-_3 \\
\end{array}
\]

Construction 2

Let \(p \equiv 1 \text{ or } 9 \pmod{40}\), \(R = \mathbb{Z}_p^2\), and let \(\sigma \in Z(GL(2,p))\) with \(8 \mid o(\sigma)\). Let \(H = \langle \sigma, SL(2,5) \rangle\), \(M = \langle \sigma^2, SL(2,5) \rangle\).
Construction 1

1. $\langle \sigma^2 \rangle$-orbits on $R^\#$: $\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ where $(\Delta_i^+)^\sigma = \Delta_i^-$ for each $i$;
2. $S = \bigcup_{i=1}^r \Delta_i^{\epsilon_i}$ where $\epsilon_i = +$ or $-$.

Construction 2

Let $p \equiv 1$ or $9 \pmod{40}$, $R = \mathbb{Z}_p^2$, and let $\sigma \in \mathbb{Z}(\text{GL}(2, p))$ with $8 \mid o(\sigma)$. Let $H = \langle \sigma, \text{SL}(2, 5) \rangle$, $M = \langle \sigma^2, \text{SL}(2, 5) \rangle$.

1. $M$-orbits on $R^\#$: $\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-$ where $(\Delta_i^+)^\sigma = \Delta_i^-$ for each $i$;
2. $S = \bigcup_{i=1}^r \Delta_i^{\epsilon_i}$ where $\epsilon_i = +$ or $-$. 

Construction 1

1. \( \langle \sigma^2 \rangle \)-orbits on \( R^\# \): \( \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^- \) where 
\[ (\Delta_i^+)\sigma = \Delta_i^- \] for each \( i \);

2. \( S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i} \) where \( \varepsilon_i = + \) or \(-\).

Construction 2

Let \( p \equiv 1 \) or \( 9 \) (mod 40), \( R = \mathbb{Z}_p^2 \), and let \( \sigma \in \mathbb{Z}(\text{GL}(2, p)) \) with \( 8 \mid o(\sigma) \). Let \( H = \langle \sigma, \text{SL}(2, 5) \rangle \), \( M = \langle \sigma^2, \text{SL}(2, 5) \rangle \).

1. \( M \)-orbits on \( R^\# \): \( \Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^- \) where 
\[ (\Delta_i^+)\sigma = \Delta_i^- \] for each \( i \);

2. \( S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i} \) where \( \varepsilon_i = + \) or \(-\).

\( \Gamma = \text{Cay}(R, S) \) is an SC-metacirculant with \( \text{Aut}\Gamma \) insoluble.
Construction 1

1. \(\langle \sigma^2 \rangle\)-orbits on \(R^#\): \(\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-\) where 
   \((\Delta_i^+)\sigma = \Delta_i^-\) for each \(i\);
2. \(S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}\) where \(\varepsilon_i = +\) or \(-\).

Construction 2

Let \(p \equiv 1\) or \(9\) (mod 40), \(R = \mathbb{Z}_p^2\), and let \(\sigma \in \mathbb{Z}(\text{GL}(2, p))\) with 
\(8 \mid o(\sigma)\). Let \(H = \langle \sigma, \text{SL}(2, 5) \rangle\), \(M = \langle \sigma^2, \text{SL}(2, 5) \rangle\).

1. \(M\)-orbits on \(R^#\): \(\Delta_1^+, \Delta_1^-, \Delta_2^+, \Delta_2^-, \ldots, \Delta_r^+, \Delta_r^-\) where 
   \((\Delta_i^+)\sigma = \Delta_i^-\) for each \(i\);
2. \(S = \bigcup_{i=1}^r \Delta_i^{\varepsilon_i}\) where \(\varepsilon_i = +\) or \(-\).

Lemma 3

There exist SC-metacirculants of order \(p_1^2 \ldots p_t^2\) (\(p_i\) distinct) with 
\(\text{Aut}\Gamma \geq \mathbb{Z}_p^2 \cdots \mathbb{Z}_p^t : (\mathbb{Z}_\ell \circ \text{SL}(2, 5)^t)\).
Theorem 4 (Li and Praeger, 2012)

The automorphism group of a self-complementary circulant is soluble.
Self-complementary metacirculants

Theorem 4 (Li and Praeger, 2012)

The automorphism group of a self-complementary circulant is soluble.

The following theorem extends the result.

Theorem 5 (Li, Rao and Song, 2014)

The automorphism group of a self-complementary metacirculant is either soluble, or contains composition factor $A_5$. 
Proof of theorem 5

- $\Gamma$ is a self-complementary metacirculant
- $\sigma$ is a complementing isomorphism
- $G = \text{Aut}\Gamma$
- $X = \langle G, \sigma \rangle = G.\mathbb{Z}_2$
- $R < G$ is transitive and metacyclic
Proof of theorem 5

- $\Gamma$ is a self-complementary metacirculant
- $\sigma$ is a complementing isomorphism
- $G = \text{Aut}\Gamma$
- $X = \langle G, \sigma \rangle = G\mathbb{Z}_2$
- $R < G$ is transitive and metacyclic

A block system $\mathcal{B}$ is a nontrivial $X$-invariant partition of $V$.

(i) $V$ has no block systems $\Rightarrow X$ is primitive.
(ii) $V$ has a block system $\Rightarrow X$ is imprimitive.
The primitive case

**Theorem 6 (Guralnick, Li, Praeger and Saxl, 2004)**

*If $X$ is primitive, then*

(i) $X$ is affine, or

(ii) $X$ is of product action type with $\text{soc}(X) = \text{PSL}(2, q^2)^\ell$. 
The primitive case

Theorem 6 (Guralnick, Li, Praeger and Saxl, 2004)

If $X$ is primitive, then

(i) $X$ is affine, or

(ii) $X$ is of product action type with $\text{soc}(X) = \text{PSL}(2, q^2)^\ell$.

1. $R$ is metacyclic $\Rightarrow$ $X$ is affine of dimension $\leq 2$. 
The primitive case

Theorem 6 (Guralnick, Li, Praeger and Saxl, 2004)

If $X$ is primitive, then

(i) $X$ is affine, or

(ii) $X$ is of product action type with $\text{soc}(X) = \text{PSL}(2, q^2)^\ell$.

1. $R$ is metacyclic $\Rightarrow$ $X$ is affine of dimension $\leq 2$.
2. $X$ is insoluble $\Rightarrow$ $X = \mathbb{Z}_p^2 : (\mathbb{Z}_\ell \circ \text{SL}(2, 5))$ ▶ construction
The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G^B_B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G^B \leq \text{Aut}\Sigma$ and each element of $X^B \setminus G^B$ is its complementing isomorphism.
The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G^B_B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G^B \leq \text{Aut}\Sigma$ and each element of $X^B \setminus G^B$ is its complementing isomorphism.

1. Let $B$ be a minimal block system of $V$. 
The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G^B_B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G^B \leq \text{Aut}\Sigma$ and each element of $X^B \setminus G^B$ is its complementing isomorphism.

1. Let $B$ be a minimal block system of $V$.
2. Then $X = K.X^B$, and $X^B_B$ primitive on $B$. 
The imprimitive case

Theorem 7 (Li and Praeger, 2003)
If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G^B_B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G^B \leq \text{Aut}\Sigma$ and each element of $X^B \setminus G^B$ is its complementing isomorphism.

1. Let $B$ be a minimal block system of $V$.
2. Then $X = K.X^B$, and $X^B_B$ primitive on $B$.
3. $R^B_B < X^B_B \Rightarrow$ If $X^B_B$ insoluble, then $X^B_B = \mathbb{Z}_p^2:(\mathbb{Z}_\ell \circ \text{SL}(2, 5))$. 
The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G_B^B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G_B^B \leq \text{Aut}\Sigma$ and each element of $X_B^B \setminus G_B^B$ is its complementing isomorphism.

1. Let $B$ be a minimal block system of $V$.
2. Then $X = K.X_B^B$, and $X_B^B$ primitive on $B$.
3. $R_B^B < X_B^B \Rightarrow$ If $X_B^B$ insoluble, then $X_B^B = \mathbb{Z}_p^2:(\mathbb{Z}_\ell \circ \text{SL}(2, 5))$.
4. Consider $K_B^B \leq X_B^B$ and $K \leq K_B^{B_1} \times \ldots \times K_B^{B_2}$. 
The imprimitive case

Theorem 7 (Li and Praeger, 2003)

If $X$ is imprimitive, then:

(i) $[B]_\Gamma$ is self-complementary, $G_B \leq \text{Aut}[B]_\Gamma$, and $\sigma^B$ is its complementing isomorphism;

(ii) there is a self-complementary graph $\Sigma$ with vertex set $B$ such that $G^B \leq \text{Aut}\Sigma$ and each element of $X^B \setminus G^B$ is its complementing isomorphism.

1. Let $B$ be a minimal block system of $V$.
2. Then $X = K.X^B$, and $X^B_B$ primitive on $B$.
3. $R^B_B < X^B_B \Rightarrow$ If $X^B_B$ insoluble, then $X^B_B = \mathbb{Z}_p^2:(\mathbb{Z}_\ell \circ \text{SL}(2, 5))$.
4. Consider $K^B \leq X^B_B$ and $K \leq K^{B_1} \times \ldots \times K^{B_2}$.
5. $R^B \leq X^B$ is transitive and metacyclic. \hfill $\square$
Conjecture 8

Self-complementary metacirculants are Cayley graphs?  ▶ Example
Thank you!