# Monodromy Groups of Polytopes

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(from projects with L.Berman, D.Oliveros, E.Schulte and G.Williams)

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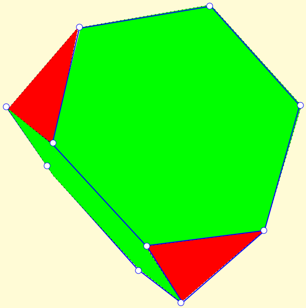
# Regularity is rare, despite its ubiquity

An $n$-polytope $\mathcal{P}$ is regular if $\mathrm{Aut}(\mathcal{P})$ is transitive on flags.
But most polytopes of rank $n \geq 3$ are not regular.

Eg. The truncated tetrahedron $\mathcal{Q}$, although quite symmetrical, has facets of two types (and 3 flag orbits under the action of $\mathrm{Aut}(\mathcal{Q}) \simeq S_4$).



# Now lift to covers ...

- Likewise, a map $\mathcal{Q}$ on a compact surface will not usually be regular.
- But it is well-known that $\mathcal{Q}$ is covered by a regular map $\mathcal{P}$ (usually on some other surface).
- The regular cover $\mathcal{P}$ is unique (to isomorphism) if it covers $\mathcal{Q}$ minimally.

- The proof is straightforward and works for any abstract 3-polytope (e.g. if $\mathcal{Q}$ is a face-to-face tessellation of the plane). In fact,

$$\mathrm{Aut}(\mathcal{P}) \simeq \mathrm{Mon}(\mathcal{Q}), \text{ the monodromy group of } \mathcal{Q}\,.$$

- So it's crucial that $\mathrm{Mon}(\mathcal{Q})$ **is** a string C-group when rank $n = 3$.

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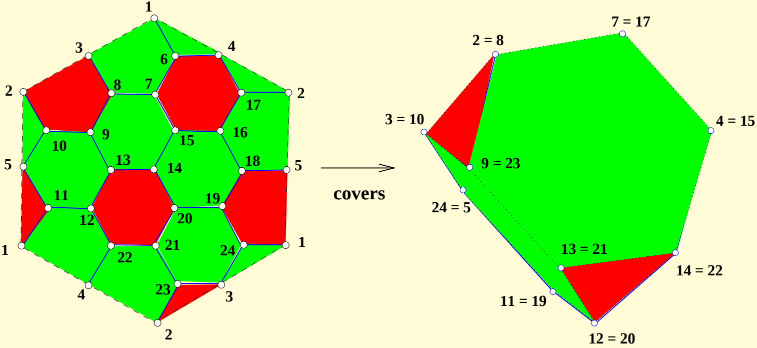
# Example.

Hartley and Williams (2009) determined the minimal regular cover $\mathcal{P}$ for each classical (convex) Archimedean solid $\mathcal{Q}$ in $\mathbb{E}^3$.

Here the regular toroidal map $\mathcal{P} = \{6,3\}_{(2,2)}$ covers the truncated tetrahedron $\mathcal{Q}$.



# More generally, L. Berman, D. Oliveros, G. Williams and I

now have

**Theorem** (on the front burner). For $n \geq 2$, let $M_n = \langle r_0, r_1, \ldots, r_{n-1} \rangle$ be the monodromy group of the truncated $n$-simplex. Then
(a) $M_n$ is a string C-group of type $\{6, 3, \ldots, 3\}$.
(b) $M_n$ is isomorphic to $S_{n+1} \times S_n$.
(c) A presentation for $M_n$ comes from adjoining to the standard relations for Coxeter group with diagram $\bullet \overset{6}{—} \bullet — \bullet - \cdots - \bullet — \bullet$ (on $n$ nodes) just one extra magic relation:

$$(r_0 r_1 r_0 r_1 r_2)^4 = e.$$

(for $n \geq 3$). This relation is independent of rank.

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# This means

(a) For $n \geq 1$, the truncated $n$-simplex has an essentially unique minimal regular cover $\mathcal{P}_n$ with $n!\,(n+1)!$ flags.

(b) For $n \geq 4$, $\mathcal{P}_n$ is the universal regular polytope for facets of type $\mathcal{P}_{n-1}$ and simplicial vertex-figures.

(c) $M_n$ is a mix of the sort described in [ARP, 7A12].

(d) And there are related finite, regular polytopes of types $\{6, 3, \ldots, 3\}$, $\{3, 6, 3, \ldots, 3\}$, $\{6, 3, 6, \ldots, 3\}$, etc.

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# Let's continue with covers in general:

- every polytope of small rank $n \leq 2$ is (combinatorially=abstractly) regular, hence equals its own minimal regular cover.
- every (abstract) 3-polytope $\mathcal{Q}$ has a unique minimal regular cover $\mathcal{P}$, and $\mathrm{Mon}(\mathcal{Q}) \simeq \mathrm{Aut}(\mathcal{P})$.
- So it's clear (in rank $n = 3$) that the cover $\mathcal{P}$ is finite if-f $\mathcal{Q}$ is finite.
- On the other hand, any polytope in any rank $n \geq 2$ is covered by the universal regular $n$-polytope $\mathcal{U} = \{\infty, \ldots, \infty\}$.
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The natural tool $\mathrm{Mon}(Q)$ might fail the needs of polytopality.

Recently, Egon Schulte and I found a fix. From this we are able to prove, for the first time,

**Theorem** (2013, to appear in J. Alg. Comb.)
Every finite $n$-polytope $\mathcal{Q}$ is covered by a finite regular $n$-polytope $\mathcal{P}$. Moreover, if $\mathcal{Q}$ has all its $k$-faces isomorphic to one particular regular $k$-polytope $\mathcal{K}$, then we may choose $\mathcal{P}$ to also have such $k$-faces.

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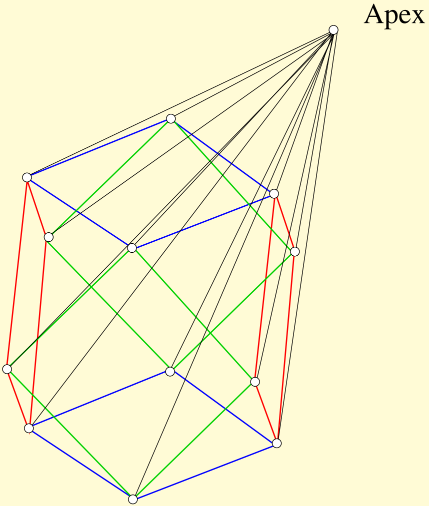
# A 4-dimensional convex example

Suppose $\mathcal{Q}$ is the pyramid over a cuboctahedral base.
Then from our theorem, $\mathcal{Q}$ has a regular cover $\mathcal{P}$ of type $\{12, 12, 12\}$ and with

$$2^{53} \cdot 3^{14} \cdot 5 \approx 2.15 \times 10^{23}$$

flags. (This isn't likely a minimal cover!)



# Idea of proof.

- an induction based on rank of regular initial sections in $\mathcal{Q}$
- crucial case is when $n$-polytope $\mathcal{Q}$ has all facets isomorphic to some regular $(n-1)$-polytope $\mathcal{K}$
- in that case, extend $\mathcal{K}$ 'trivially' to a regular $n$-polytope $\bar{\mathcal{K}}$ of type $\{\mathcal{K}, 2\}$... Thanks ...
- next 'mix' to get

$$G = \mathrm{Mon}(\mathcal{Q}) \Diamond \mathrm{Aut}(\bar{\mathcal{K}})$$

- then $G = \mathrm{Aut}(\mathcal{P})$ for desired regular cover $\mathcal{P}$ of $\mathcal{Q}$ (quotient criterion).
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# References

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[2] P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Encyclopedia of Mathematics and its Applications, **92**, Cambridge University Press, Cambridge, 2002.

[3] B. Monson and E. Schulte, *Finite Polytopes have Finite Regular Covers*, Journal of Algebraic Combinatorics, 2013.

[4] B.Monson, D. Pellicer and G. Williams, *Mixing and Monodromy of Abstract Polytopes*, Trans. AMS., 2014.

# Exercise: prove (if you didn't know it):

For $p \geq 2$, the Coxeter group of rank $n$ and diagram

$$\bullet\overset{2p}{\rule{2em}{0.4pt}}\bullet\overset{3}{\rule{2em}{0.4pt}}\bullet - \cdots - \bullet\overset{3}{\rule{2em}{0.4pt}}\bullet$$

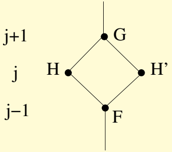
has a subgroup of index $\left( \begin{array}{c} n \\ j+1 \end{array} \right)$ which is isomorphic in turn to the Coxeter group with diagram

$$\bullet\overset{3}{\rule{2em}{0.4pt}}\bullet - \cdots - \bullet\overset{3}{\rule{2em}{0.4pt}}\bullet\overset{2p}{\rule{2em}{0.4pt}}\bullet\overset{p}{\rule{2em}{0.4pt}}\bullet\overset{2p}{\rule{2em}{0.4pt}}\bullet\overset{3}{\rule{2em}{0.4pt}}\bullet - \cdots - \bullet\overset{3}{\rule{2em}{0.4pt}}\bullet\,,$$

where the first "$2p$" labels the $j$ th branch of the diagram.

# What is the monodromy group of an $n$-polytope $\mathcal{P}$?

Any $n$-polytope $\mathcal{P}$ (abstract, convex, ...) satisfies the *diamond property*: whenever $F < G$ with $\text{rank}(F) = j - 1$ and $\text{rank}(G) = j + 1$ then there exist exactly two $j$-faces $H$ with $F < H < G$




So each flag $\Phi$ in $\mathcal{P}$ is $j$-adjacent to a unique flag $\Phi^j$. Since $(\Phi^j)^j = \Phi$, the mapping $r_j : \Phi \mapsto \Phi^j$ is a fixed-point free involution on the set $\mathcal{F}(\mathcal{P})$ of all flags of $\mathcal{P}$.

The monodromy group $\text{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$ is then a subgroup of the symmetric group on $\mathcal{F}(\mathcal{P})$.

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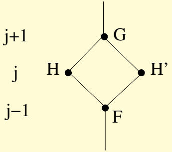


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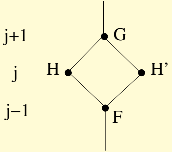


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# What are abstract polytopes?

An **abstract *n*-polytope** $\mathcal{Q}$ is a poset having some of the key structural properties of the face lattice of a convex *n*-polytope, although $\mathcal{Q}$

- need not be a lattice
- need not be finite
- need not have a familiar geometric realization

The abstract 3-polytopes include all convex polyhedra, face-to-face tessellations and many less familiar structures. But

you can safely think of a finite 3-polytope as a *map on a compact surface*.

▸ Do we want details?

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you can safely think of a finite 3-polytope as a *map on a compact surface*.

Do we want details?

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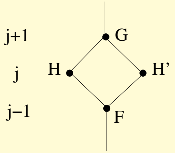


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  via adjacency in the flag graph; this rules out, for example, the disjoint union of two polyhedra
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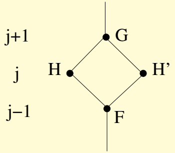


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# The symmetry of $\mathcal{Q}$

is encoded in the group $\Gamma = \Gamma(\mathcal{Q})$ of all order-preserving bijections (= automorphisms) of $\mathcal{Q}$.

Each automorphism is det'd by its action on any one *flag* $\Phi$; for a polyhedron, a flag

$$\Phi = \text{incident [vertex, edge, facet] triple}$$

Def. $\mathcal{Q}$ is *regular* if $\Gamma$ is transitive on flags.

Examples:

- any polygon ($n = 2$) is (abstractly, i.e. combinatorially) regular
- the usual tiling of $\mathbb{E}^3$ by unit cubes is an infinite regular 4-polytope
- the Platonic solids ($n = 3$).

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# The convex regular polyhedra (=Platonic solids) and the Kepler-Poinsot star-polyhedra $\mathcal{P}$

Local data for both polyhedron $\mathcal{P}$ and its group $\Gamma(\mathcal{P})$ reside in the Schläfli symbol or type $\{p, q\}$.

Platonic solids: $\{3, 3\}$ (tetrahedron), $\{3, 4\}$ (octahedron), $\{4, 3\}$ (cube), $\{3, 5\}$ (icosahedron),$\{5, 3\}$ (dodecahedron)

Kepler (ca. 1619) $\{\frac{5}{2}, 5\}$ (small stellated dodecahedron), $\{\frac{5}{2}, 3\}$ (great stellated dodecahedron)

Poinsot (ca. 1809) $\{5, \frac{5}{2}\}$ (great dodecahedron), $\{3, \frac{5}{2}\}$ (great isosahedron)

# The classical convex regular polytopes, their Schläfli symbols and finite Coxeter groups with string diagrams

| name | symbol | # facets | (Coxeter) group | order |
|---|---|---|---|---|
| $n = 4$: | | | | |
| simplex | $\{3,3,3\}$ | 5 | $A_4 \simeq S_5$ | 5! |
| cross-polytope | $\{3,3,4\}$ | 16 | $B_4$ | 384 |
| cube | $\{4,3,3\}$ | 8 | $B_4$ | 384 |
| 24-cell | $\{3,4,3\}$ | 24 | $F_4$ | 1152 |
| 600-cell | $\{3,3,5\}$ | 600 | $H_4$ | 14400 |
| 120-cell | $\{5,3,3\}$ | 120 | $H_4$ | 14400 |
| $n > 4$: | | | | |
| simplex | $\{3,3,\ldots,3\}$ | $n+1$ | $A_n \simeq S_{n+1}$ | $(n+1)!$ |
| cross-polytope | $\{3,\ldots,3,4\}$ | $2^n$ | $B_n$ | $2^n \cdot n!$ |
| cube | $\{4,3,\ldots,3\}$ | $2n$ | $B_n$ | $2^n \cdot n!$ |

# Regular polytopes and string C-groups

Schulte (1982) showed that the abstract regular $n$-polytopes $\mathcal{P}$ correspond exactly to the *string C-groups of rank n* (which we often study in their place).

**The Correspondence Theorem**.

**Part 1**. If $\mathcal{P}$ is a regular $n$-polytope, then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a *string C-group*.

**Part 2**. Conversely, if $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group, then we can reconstruct an $n$-polytope $\mathcal{P}(\Gamma)$ (in a natural way as a coset geometry on $\Gamma$).

Furthermore, $\Gamma(\mathcal{P}(\Gamma)) \simeq \Gamma$ and $\mathcal{P}(\Gamma(\mathcal{P})) \simeq \mathcal{P}$.

# Recap: what is a string C-group?

**Means**: having fixed a base flag $\Phi$ in $\mathcal{P}$, for $0 \leq j \leq n-1$ there is a unique automorphism $\rho_j \in \Gamma(\mathcal{P})$ mapping $\Phi$ to the $j$-adjacent flag $\Phi^j$. These involutions generate $\Gamma(\mathcal{P})$ and satisfy the relations implicit in some string (Coxeter) diagram, like

$$\bullet\overset{p_1}{\rule{2em}{0.4pt}}\bullet\overset{p_2}{\rule{2em}{0.4pt}}\bullet\rule{1em}{0.4pt}\ \ldots\ \rule{1em}{0.4pt}\bullet\overset{p_{n-1}}{\rule{2em}{0.4pt}}\bullet\ ,$$

and perhaps other relations, so long as this *intersection condition* continues to hold:

$$\langle \rho_k : k \in I \rangle \cap \langle \rho_k : k \in J \rangle = \langle \rho_k : k \in I \cap J \rangle$$

(for all $I, J \subseteq \{0, \ldots, n-1\}$).
Notice that $\mathcal{P}$ then has Schläfli type $\{p_1, \ldots, p_{n-1}\}$.