Mirror Automorphisms of Regular Maps

Adnan Melekoğlu

Adnan Menderes University
Aydın, TURKEY

SIGMAP 2014
West Malvern, 7-11 July 2014

Regular Maps and Riemann surfaces

A compact Riemann surface X of genus g is called symmetric if it admits an anti-conformal involution $\sigma\colon X \to X$, which is called a symmetry of X.

Regular Maps and Riemann surfaces

A compact Riemann surface X of genus g is called symmetric if it admits an anti-conformal involution $\sigma\colon X \to X$, which is called a symmetry of X.

The fixed-point set of σ is either empty or consists disjoint simple closed geodesics on X, which are called the mirrors of σ.

Regular Maps and Riemann surfaces

A compact Riemann surface X of genus g is called symmetric if it admits an anti-conformal involution $\sigma \colon X \to X$, which is called a symmetry of X.

The fixed-point set of σ is either empty or consists disjoint simple closed geodesics on X, which are called the mirrors of σ.

A map $\mathcal{M}$ on a Riemann surface X is an embedding of a finite connected graph $\mathcal{G}$ into X such that the components of $X - \mathcal{G}$ are open discs, which are called the faces of $\mathcal{M}$.

Regular Maps and Riemann surfaces

A compact Riemann surface X of genus g is called symmetric if it admits an anti-conformal involution $\sigma \colon X \to X$, which is called a symmetry of X.

The fixed-point set of σ is either empty or consists disjoint simple closed geodesics on X, which are called the mirrors of σ.

A map $\mathcal{M}$ on a Riemann surface X is an embedding of a finite connected graph $\mathcal{G}$ into X such that the components of $X - \mathcal{G}$ are open discs, which are called the faces of $\mathcal{M}$.

$\mathcal{M}$ is said to be of type (m, n) if every vertex and face of $\mathcal{M}$ has valency m and n respectively.

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$Aut^+\mathcal{M}$: Group of conformal automorphisms of $\mathcal{M}$

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$Aut^{+}\mathcal{M}$: Group of conformal automorphisms of $\mathcal{M}$

$Aut^{\pm}\mathcal{M}$: Group of all automorphisms of $\mathcal{M}$

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$Aut^{+}\mathcal{M}$: Group of conformal automorphisms of $\mathcal{M}$

$Aut^{\pm}\mathcal{M}$: Group of all automorphisms of $\mathcal{M}$

$\mathcal{M}$ is said to be regular if $Aut^{+}\mathcal{M}$ is transitive on the directed edges.

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$Aut^{+}\mathcal{M}$: Group of conformal automorphisms of $\mathcal{M}$
$Aut^{\pm}\mathcal{M}$: Group of all automorphisms of $\mathcal{M}$

$\mathcal{M}$ is said to be regular if $Aut^{+}\mathcal{M}$ is transitive on the directed edges.

A reflection of $\mathcal{M}$ is a symmetry of X that fixes some mirrors.

Regular Maps and Riemann surfaces

An automorphism of $\mathcal{M}$ is an automorphism of X that leaves $\mathcal{M}$ invariant and preserves incidence.

$Aut^{+}\mathcal{M}$: Group of conformal automorphisms of $\mathcal{M}$

$Aut^{\pm}\mathcal{M}$: Group of all automorphisms of $\mathcal{M}$

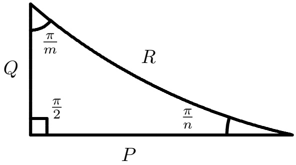
$\mathcal{M}$ is said to be regular if $Aut^{+}\mathcal{M}$ is transitive on the directed edges.

A reflection of $\mathcal{M}$ is a symmetry of X that fixes some mirrors.

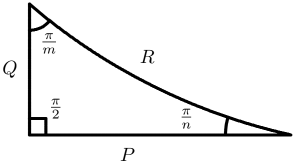
All mirrors on X fixed by the reflections of $\mathcal{M}$ divide X into triangles with angles $\pi/2$, π/m and π/n. Each of these triangles is called a $(2, m, n)$-triangle.

Regular Maps and Riemann surfaces

Let T be a $(2, m, n)$-triangle on X and let P, Q and R denote the reflections in the sides of T.



Let T be a $(2, m, n)$-triangle on X and let P, Q and R denote the reflections in the sides of T.



These reflections generate $Aut^{\pm}\mathcal{M}$ and can be chosen in order that they satisfy the relations

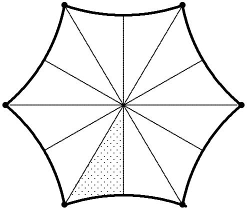
$$P^2 = Q^2 = R^2 = (PQ)^2 = (QR)^m = (RP)^n = 1. \tag{1}$$

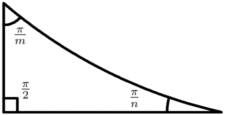
Patterns of Mirrors

The geometric points of $\mathcal{M}$ are the vertices, the face-centers and the edge-centers. So each geometric point is a corner of a $(2, m, n)$-triangle.

Patterns of Mirrors

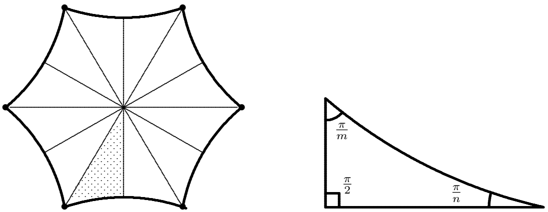
The geometric points of $\mathcal{M}$ are the vertices, the face-centers and the edge-centers. So each geometric point is a corner of a $(2, m, n)$-triangle.





Patterns of Mirrors

The geometric points of $\mathcal{M}$ are the vertices, the face-centers and the edge-centers. So each geometric point is a corner of a $(2, m, n)$-triangle.



Every mirror M of a reflection of $\mathcal{M}$ passes through some of the geometric points of $\mathcal{M}$ such that these points form a periodic sequence, which is called the pattern of M.

Patterns of Mirrors

F. Klein, *Über die Transformation siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1879), 428–471.

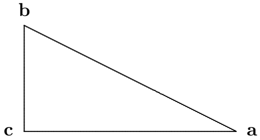
Patterns of Mirrors

F. Klein, *Über die Transformation siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1879), 428–471.

b : vertex

c : edge center

a : face center



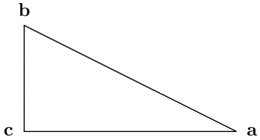
Patterns of Mirrors

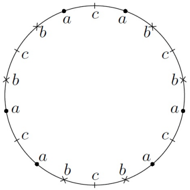
F. Klein, *Über die Transformation siebenter Ordnung der elliptischen Funktionen*, Math. Ann. **14** (1879), 428–471.

b : vertex

c : edge center

a : face center





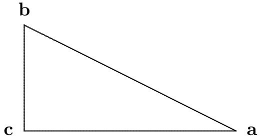
abcbac abcbac abcbac

Patterns of Mirrors

b : vertex

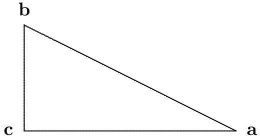
c : edge center

a : face center



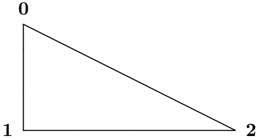
Patterns of Mirrors

b : vertex
c : edge center
a : face center



H.S.M. Coxeter, *Regular Polytopes*, Dover, 1973.

0 : vertex
1 : edge center
2 : face center



Patterns of Mirrors

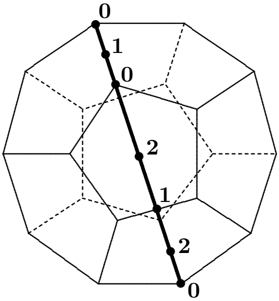
Example

Every mirror on the sphere fixed by a reflection of the map $(3, 5)$ has pattern $\mathbf{010212010212}$ which we abbreviate to $(\mathbf{010212})^2$.

Patterns of Mirrors

Example

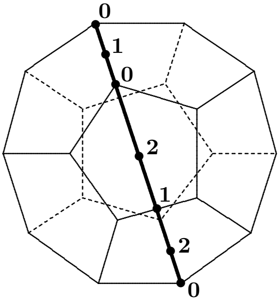
Every mirror on the sphere fixed by a reflection of the map $(3,5)$ has pattern **010212010212** which we abbreviate to $(\mathbf{010212})^2$.



Patterns of Mirrors

Example

Every mirror on the sphere fixed by a reflection of the map $(3,5)$ has pattern **010212010212** which we abbreviate to $(\mathbf{010212})^2$.



Here **010212** is called a link of the pattern.

Patterns of Mirrors

Case	Reflections	Pattern
m and n odd	P, Q, R	$(\mathbf{010212})^K$
m odd n even	P	$(\mathbf{12})^K$
	Q, R	$(\mathbf{0102})^K$
m and n even	P	$(\mathbf{12})^K$
	Q	$(\mathbf{01})^K$
	R	$(\mathbf{02})^K$
m even n odd	P, R	$(\mathbf{0212})^K$
	Q	$(\mathbf{01})^K$

Patterns of Mirrors

Case	Reflections	Pattern
m and n odd	P, Q, R	$(\mathbf{010212})^K$
m odd n even	P	$(\mathbf{12})^K$
	Q, R	$(\mathbf{0102})^K$
m and n even	P	$(\mathbf{12})^K$
	Q	$(\mathbf{01})^K$
	R	$(\mathbf{02})^K$
m even n odd	P, R	$(\mathbf{0212})^K$
	Q	$(\mathbf{01})^K$

There are six possible patterns: $(\mathbf{01})^K$, $(\mathbf{02})^K$, $(\mathbf{12})^K$, $(\mathbf{0102})^K$, $(\mathbf{0212})^K$, $(\mathbf{010212})^K$.

Mirror Automorphisms

Let $\mathcal{M}$ be a regular map on a compact Riemann surface X and let M be a mirror of a reflection of $\mathcal{M}$. Then there exist two particular conformal automorphisms of $\mathcal{M}$ that fix M setwise and rotate it in opposite directions.

Mirror Automorphisms

Let $\mathcal{M}$ be a regular map on a compact Riemann surface X and let M be a mirror of a reflection of $\mathcal{M}$. Then there exist two particular conformal automorphisms of $\mathcal{M}$ that fix M setwise and rotate it in opposite directions.

These automorphisms are inverses of each other and they are called the mirror automorphisms of M.

Mirror Automorphisms

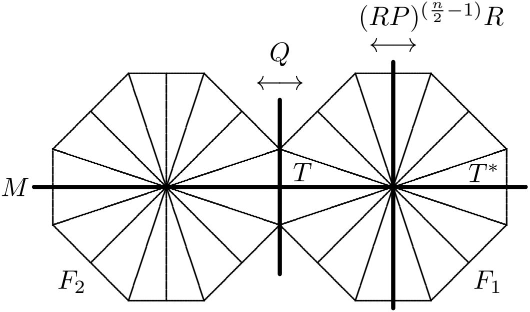
Let $\mathcal{M}$ be a regular map on a compact Riemann surface X and let M be a mirror of a reflection of $\mathcal{M}$. Then there exist two particular conformal automorphisms of $\mathcal{M}$ that fix M setwise and rotate it in opposite directions.

These automorphisms are inverses of each other and they are called the mirror automorphisms of M.

Sometimes a mirror may have only one mirror automorphism in which case the mirror automorphism is the identity or has order 2.

Mirror Automorphisms

Mirror automorphism of a mirror with pattern $(\mathbf{12})^K$:



$(RP)^{(\frac{n}{2}-1)}RQ$ is a rotary automorphism for M

Mirror Automorphisms

Patterns and Mirror Automorphisms:

Case	Pattern Link	Mirror Automorphism
m and n even	**12**	$(RP)^{(\frac{n}{2}-1)}RQ$
	01	$(RQ)^{(\frac{m}{2}-1)}RP$
	02	$(PR)^{(\frac{n}{2}-1)}P(QR)^{(\frac{m}{2}-1)}Q$
m and n have different parities	**0102**	$(PR)^{(\frac{n}{2}-1)}P(QR)^{\frac{m-1}{2}}P(RQ)^{\frac{m-1}{2}}$
	0212	$(PR)^{\frac{n-1}{2}}Q(RP)^{\frac{n-1}{2}}(QR)^{(\frac{m}{2}-1)}Q$
m and n odd	**010212**	$(QR)^{\frac{m-1}{2}}P(RQ)^{\frac{m-1}{2}}(PR)^{\frac{n-1}{2}}Q(RP)^{\frac{n-1}{2}}$

Wiman surfaces

The order of an automorphism of a Riemann surface X of genus $g > 1$ cannot exceed $4g+2$. In this case $Aut^+X \simeq C_{4g+2}$ and X corresponds to a normal subgroup of the triangle group [2,2g+1,4g+2].

Wiman surfaces

The order of an automorphism of a Riemann surface X of genus $g > 1$ cannot exceed $4g+2$. In this case $Aut^{+}X \simeq C_{4g+2}$ and X corresponds to a normal subgroup of the triangle group [2,2g+1,4g+2].

The second maximum possible order is $4g$ and in this case $|Aut^{+}X| = 8g$ and X corresponds to a normal subgroup of the triangle group $[2, 4, 4g]$.

Wiman surfaces

The order of an automorphism of a Riemann surface X of genus $g > 1$ cannot exceed $4g+2$. In this case $Aut^{+}X \simeq C_{4g+2}$ and X corresponds to a normal subgroup of the triangle group [2,2g+1,4g+2].

The second maximum possible order is $4g$ and in this case $|Aut^{+}X| = 8g$ and X corresponds to a normal subgroup of the triangle group $[2, 4, 4g]$.

These surfaces are known as Wiman surfaces of type I and II respectively.

Regular Maps with Identity Mirror Automorphisms

Case	Pattern	Map type	Underlying surface
m and n even	**12**	$(2g+1, 4g+2)$, $(4g, 4g)$	Wiman I, Wiman II
	01	$(4g+2, 2g+1)$, $(4g, 4g)$	Wiman I, Wiman II
	02	$(4g, 4g)$, $(4, 4g)$	Wiman II
m and n have different parities	**0102**	$(2g+1, 4g+2)$	Wiman I
	0212	$(4g+2, 2g+1)$	Wiman I
m and n odd	**010212**	?	?

Thank you!