Are there (non-trivial) matroidal Galois invariants of dessins?

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SIGMAP 2014
West Malvern, UK

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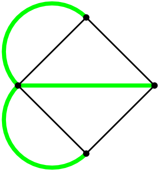

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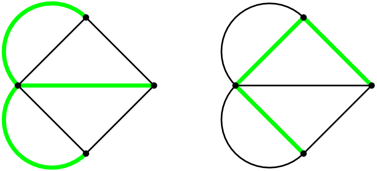

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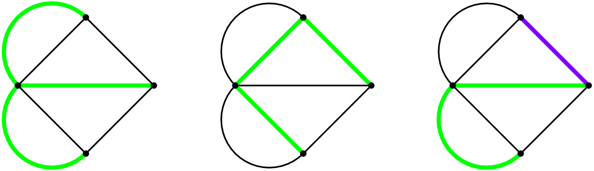

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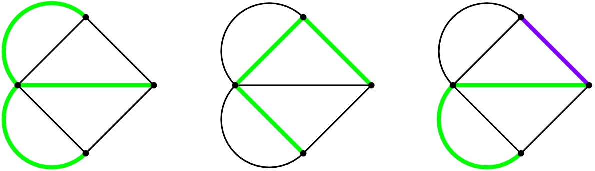

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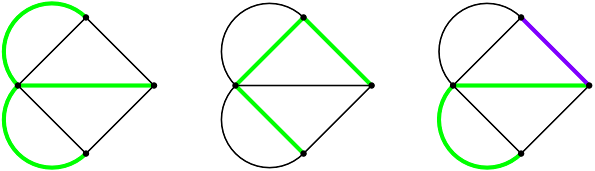

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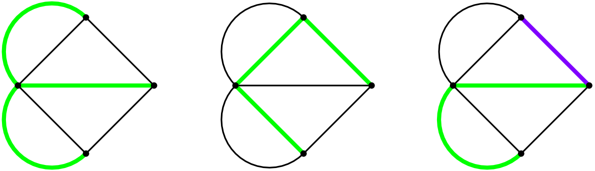

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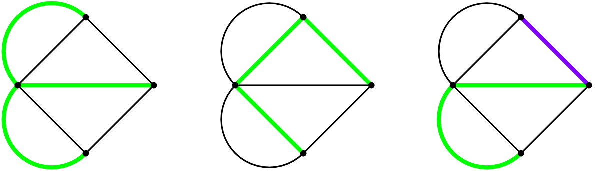

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Not every matroid is graphic.

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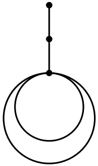

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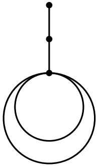

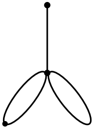

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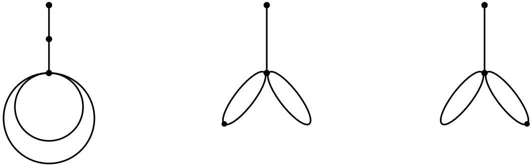

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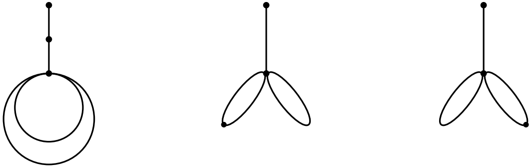

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Comparing self duality

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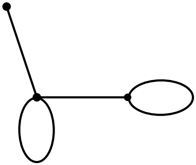

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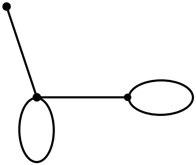

If G is 3-connected, then the three notions of self-duality coincide.

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Evidence? Brute force (Adrianov et al, *Catalog Of Dessins d'Enfants With No More Than 4 Edges*, J. of Math. Sci. 158(1)).

Questions

- Is the matroid self-duality determined by the passport? Is there an example two maps with the same passport such that one map has a self-dual matroid but the other one doesn't?

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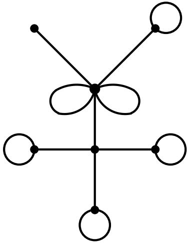
- Is there an interesting family of maps for which matroid self-duality is clearly an invariant?

$$[v_1^{m_1} \cdots v_k^{m_k}, 2^{2n}, f_1^1 1^n],$$
$$m_1 + \cdots + m_k = n + 1,$$
$$\sum m_j v_j = f_1 + n = 2 \cdot 2n.$$



Questions

- What about higher genus?

Questions

- What about higher genus?

Questions

- What about higher genus? We need to pass to Lagrangian (also known as delta or symmetric) matroids.

Thank You!