Groups of Ree type in characteristic 3 acting on polytopes

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- sporadic groups (L.–Vauthier, Hartley–Hulpke, L.–Mixer, Connor–L.–Mixer, Connor–L.)

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However, the rank 2 groups which are used to define them, do impose some structure on these sets.

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Ree groups seem to be misfits in a lot of general theories about Chevalley groups and their twisted analogues:
- no applications yet of the Curtis-Tits-Phan theory for Ree groups;
- all finite quasisimple groups of Lie type are known to be presented by two elements and 51 relations, except the Ree groups in characteristic 3 (Guralnick–Kantor–Kassasbov–Lubotsky 2011).

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Jones (1994) extended this result to arbitrary simple Ree groups $\mathrm{R}(q)$, proving in particular that the corresponding presentations give chiral maps on surfaces.
$\Rightarrow$ $\mathrm{R}(q)$ are also automorphism groups of abstract chiral polyhedra.

2. String C-groups

Definition

A **C-group** is a group G generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following property, called the **intersection property**.

$$\forall J, K \subseteq \{0, \ldots, n-1\},$$

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

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A C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a **string C-group** if its generators satisfy the following relations.

$$(\rho_j \rho_k)^2 = 1_G \forall j, k \in \{0, \ldots n-1\} with \mid j - k \mid \geq 2$$

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The **Schläfli symbol** of a string C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is the ordered sequence $\{o(\rho_0\rho_1), \ldots, o(\rho_{n-1}\rho_n)\}$ where $o(g)$ denotes the order of the element $g \in G$.

3. Ree groups $\mathrm{R}(q)$

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such that any two points of Ω lie in exactly one block.
This Steiner system is also called a *Ree unital*. In particular, G acts 2-transitively on the points and transitively on the incident pairs of points and blocks of $\mathcal{S}$.

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G has a unique conjugacy class of involutions (Ree, 1960). Every involution ρ of G has a block B of $\mathcal{S}$ as its set of fixed points, and B is invariant under the centralizer $C_G(\rho)$ of ρ in G. Moreover, $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$, where $C_2 = \langle \rho \rangle$ and the $\mathrm{PSL}_2(q)$-factor acts on the $q+1$ points in B as it does on the points of the projective line $PG(1, q)$.

3. Ree groups $\mathrm{R}(q)$

The Ree groups $\mathrm{R}(q)$ are simple except when $q = 3$.
In particular, $\mathrm{R}(3) \cong P\Gamma L_2(8) \cong \mathrm{PSL}_2(8) : C_3$ and the commutator subgroup $\mathrm{R}(3)'$ of $\mathrm{R}(3)$ is isomorphic to $\mathrm{PSL}_2(8)$.

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Maximal subgroups of G are known (Kleidman 1988).

- $N_G(A) \cong A : C_{q-1}$ (stabilizer of a point), where A is a 3-Sylow subgroup of G;
- $C_G(\rho) \cong C_2 \times \mathrm{PSL}_2(q)$ (stabilizer of a block), where $C_2 = \langle\rho\rangle$ and ρ is an involution of G;
- $\mathrm{R}(q')$ (stabilizer of a sub-unital of $\mathcal{S}$), where $(q')^p = q$ and p is a prime;
- $N_G(A_i)$, for $i = 1, 2, 3$, where A_i is a cyclic subgroup of G of one of the following kinds:
  - $A_1 = C_{\frac{q+1}{4}}$, with $N_G(A_1) \cong (C_2^2 \times D_{\frac{q+1}{2}}) : C_3$;
  - $A_2 = C_{q+1-3^{e+1}}$, with $N_G(A_2) \cong A_2 : C_6$;
  - $A_3 = C_{q+1+3^{e+1}}$, with $N_G(A_3) \cong A_3 : C_6$.

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The automorphism group $\mathrm{Aut}(\mathrm{R}(q))$ of $\mathrm{R}(q)$ is given by

$$\mathrm{Aut}(\mathrm{R}(q)) \cong \mathrm{R}(q)' : C_{2e+1},$$

so in particular $\mathrm{Aut}(\mathrm{R}(3)) \cong \mathrm{R}(3)$.

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Theorem (L. (2006))

Among the almost simple groups G with $\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$ and $q = 2^{2e+1} \neq 2$, only the Suzuki group $\mathrm{Sz}(q)$ itself is a C-group. In particular, $\mathrm{Sz}(q)$ admits a representation as a string C-group of rank 3, but not of higher rank.

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Theorem (L.–Schulte–Van Maldeghem)

Among the almost simple groups G with $\mathrm{R}(q) \leq G \leq \mathrm{Aut}(\mathrm{R}(q))$ and $q = 3^{2e+1} \neq 3$, only the Ree group $\mathrm{R}(q)$ itself is a C-group. In particular, $\mathrm{R}(q)$ admits a representation as a string C-group of rank 3, but not of higher rank. Moreover, the non-simple Ree group $\mathrm{R}(3)$ is not a C-group.

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Pick two involutions ρ_0, ρ_1 from a maximal subgroup M of G of type $N_G(A_3)$ such that $\rho_0\rho_1$ has order $q+1+3^{e+1}$, and let B_0, B_1, respectively, denote their blocks of fixed points.

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Obviously, $B_0 \cap B_1 = \emptyset$, for otherwise $\langle \rho_0, \rho_1 \rangle$ would lie in the stabilizer of a point in $B_0 \cap B_1$, which is not possible because of the order of $\rho_0\rho_1$.

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Moreover, since the orders of $\rho_0\rho_1$ and $\rho_1\rho_2$ are coprime, the intersection property must hold as well.
Thus (G, S), with $S := \{\rho_0, \rho_1, \rho_2\}$, is a string C-group of rank 3.

4. Rank ≥ 5 case

Nothing as there is no subgroup $D_{2k} \times D_{2l}$ with $k, l \geq 3$ in G.

4. Rank 4 case

Difficult to rule out.
Use the fact that $\rho_0 \in C_G(G_{01}) \setminus N_G(G_0)$.