Galois and Hypermap Operations on Dessins

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Outline of the talk

- Hypermap operations on dessins: action of $\mathrm{Out}\, F_2 \cong \mathrm{GL}_2\, \mathbb{Z}$.
- Galois operations on dessins: action of the absolute Galois group $\mathrm{Gal}\, \overline{\mathbb{Q}}$.
- Some open problems.

Dessins and permutations

A dessin $\mathcal{D}$, or equivalently a Belyĭ pair $(\mathbf{X}, \beta)$, gives rise to

- a 2-generator permutation group $G = \langle x, y \rangle$ (finite, transitive) on the set E of edges of a bipartite map (compact, oriented),
- a conjugacy class of subgroups M (stabilisers of edges) of finite index in the free group $F_2 = \langle X, Y \mid - \rangle$ of rank 2.

The action $F_2 \to G \leq \mathrm{Sym}\, E$ of F_2 on E is given by $X \mapsto x$, $Y \mapsto y$.

G is the monodromy group $\mathrm{Mon}\, \mathcal{D}$ of $\mathcal{D}$; its centraliser in $\mathrm{Sym}\, E$ is the automorphism group $\mathrm{Aut}\, \mathcal{D}$ of $\mathcal{D}$ (preserving orientation and vertex-colours).

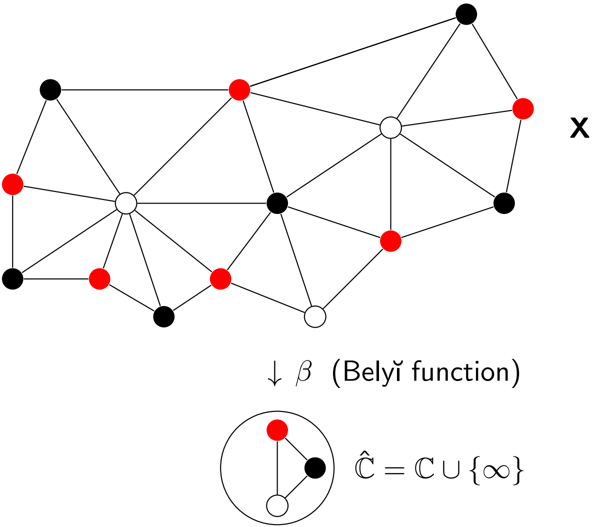


Figure : A triangulation: white, black and red vertices over $0, 1$ and ∞

(Recall Jürgen's talk.)

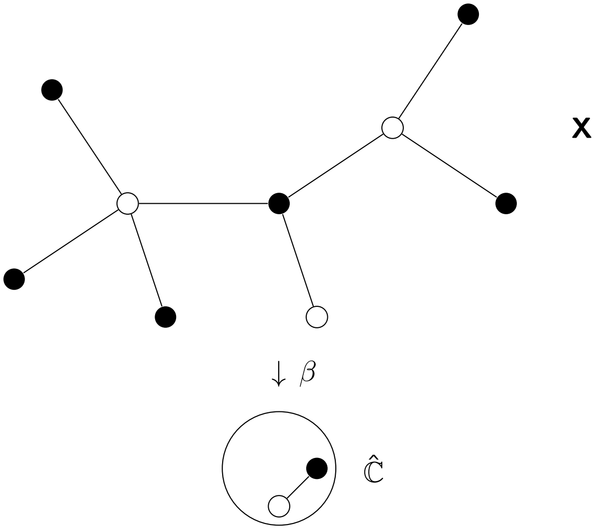


Figure : A bipartite map: white and black vertices over 0 and 1

This is the Walsh bipartite map of a hypermap (David's talk).

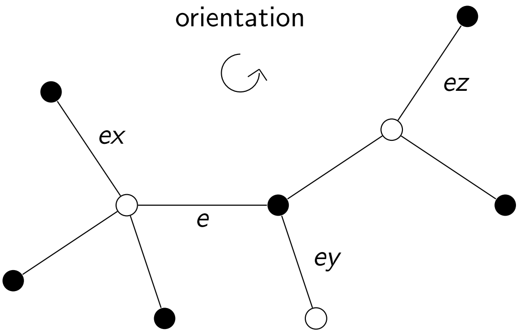


Figure : The permutations x, y and $z = (xy)^{-1}$

Here x and y rotate edges around their incident white and black vertices, following the orientation of the underlying surface **X**. (These were σ, α and $(\sigma\alpha)^{-1}$ in David's talk.)

Dessins and permutations

A dessin $\mathcal{D}$, or equivalently a Belyĭ pair $(\mathbf{X}, \beta)$, may be regarded as

- a 2-generator permutation group $G = \langle x, y \rangle$ (finite, transitive) on the set E of edges of a bipartite map (compact, oriented),
- a conjugacy class of subgroups M (stabilisers of edges) of finite index in the free group $F_2 = \langle X, Y \mid - \rangle$ of rank 2.

Here x and y rotate edges around their incident white and black vertices, following the orientation of the underlying surface $\mathbf{X}$.

Equivalently they, together with $z = (xy)^{-1}$, are the monodromy permutations (of the sheets of the covering) for the associated Belyĭ function β at the ramification points $0, 1$ and ∞ in $\hat{\mathbb{C}}$.

The universal bipartite map $\mathcal{B}_\infty$

Surface = hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.
Vertices = rationals a/b, b odd; black or white as a is even or odd.
Edge a/b to c/d (hyperbolic geodesic) iff $ad - bc = \pm 1$.
Face-centres a/b with b even (including $\infty = 1/0$).

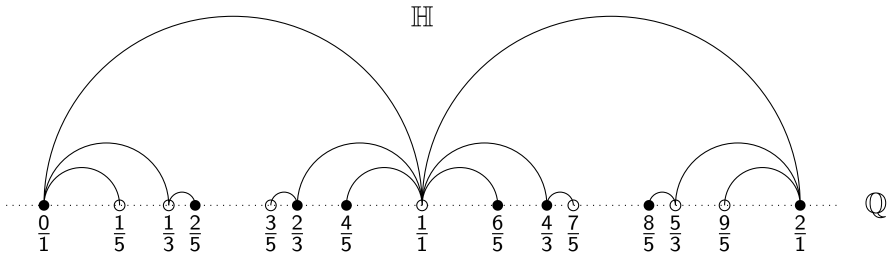
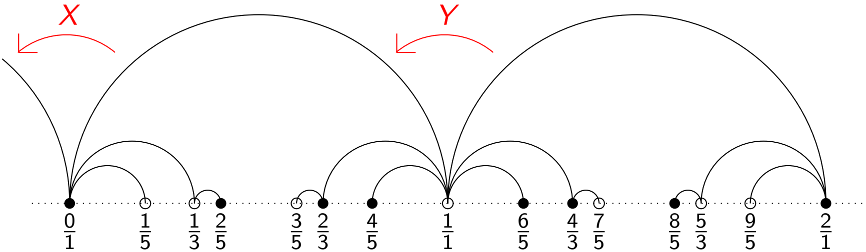


Figure : Part of $\mathcal{B}_\infty$ (for $0 \le \operatorname{Re} z \le 2$ and $b \le 5$); repeat with period 2.

Automorphisms of $\mathcal{B}_\infty$



$\mathrm{Aut}\,\mathcal{B}_\infty$ is a free group F_2 of rank 2, generated by Möbius transformations

$$X: \quad z \mapsto \frac{z}{-2z+1} \quad \text{and} \quad Y: z \mapsto \frac{z-2}{2z-3}$$

fixing the black and white vertices at 0 and 1 and cyclicly rotating their incident edges. (This is the principal congruence subgroup $\Gamma(2)$ of level 2 in the modular group $\Gamma = PSL_2\mathbb{Z}$.)

From dessins to Belyĭ pairs

Given a subgroup M of finite index in $F_2 = \mathrm{Aut}\,\mathcal{B}_\infty$, define:

- $\mathbf{X}$ to be the compactification $\overline{M\backslash\mathbb{H}}$ of the quotient surface $M\backslash\mathbb{H}$,
- $\mathcal{D}$ to be the dessin $M\backslash\mathcal{B}_\infty$ on $\mathbf{X}$,
- $\beta : \mathbf{X} \to \hat{\mathbb{C}}$ to be the projection $\overline{M\backslash\mathbb{H}} \to \overline{F_2\backslash\mathbb{H}}$ induced by the inclusion $M \leq F_2$.

Then $(\mathbf{X}, \beta)$ is the Belyĭ pair corresponding to the dessin $\mathcal{D}$ and subgroup M (up to conjugacy). Thus we have correspondences

$$(\mathbf{X}, \beta) \leftrightarrow \mathcal{D} \leftrightarrow \{M^g \mid g \in F_2\}.$$

The group Ω of hypermap operations

The group $\mathrm{Aut}\, F_2$ acts naturally on conjugacy classes of subgroups $M \le F_2$, and hence on dessins. Inner automorphisms act trivially, so there is an induced action of the outer automorphism group $\mathrm{Out}\, F_2 = \mathrm{Aut}\, F_2/\mathrm{Inn}\, F_2$ on dessins. For each $n \ge 1$ there is an epimorphism

$$\mathrm{Out}\, F_n \to \mathrm{Aut}\, (F_n^{\mathrm{ab}} = F_n/F_n' \cong \mathbb{Z}^n) = \mathrm{GL}_n\, \mathbb{Z},$$

and one can show that for $n = 2$ this is an isomorphism:

$$\mathrm{Out}\, F_2 \cong \mathrm{GL}_2\, \mathbb{Z}.$$

Thus $\mathrm{GL}_2\, \mathbb{Z}$ acts on dessins. Lynne James (EJC, 1988) showed that this action is faithful, so we obtain a group

$$\Omega \cong \mathrm{Out}\, F_2 \cong \mathrm{GL}_2\, \mathbb{Z}$$

of *hypermap operations* on dessins.

Examples of operations

The automorphism $X \leftrightarrow Y$ of F_2, corresponding to the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{Z},$$

induces the operation δ of white/black duality, transposing colours of vertices. It acts on Belyĭ pairs by $(\mathbf{X}, \beta) \leftrightarrow (\mathbf{X}, 1-\beta)$. The automorphism $X \mapsto Y \mapsto Z \mapsto X$ of F_2, corresponding to

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{Z},$$

induces a triality operation τ, permuting white and black vertices and face-centres in a 3-cycle. These operations generate the subgroup

$$\Omega_M = \langle \delta, \tau \rangle \cong S_3 \cong D_3$$

of Machì operations (Machì, Discrete Math. 1982), preserving $\mathbf{X}$.

More examples of operations

The automorphism $X \leftrightarrow X^{-1}, \ Y \leftrightarrow Y^{-1}$ of F_2, corresponding to

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{Z},$$

induces the operation $\iota : (\mathbf{X}, \beta) \mapsto (\overline{\mathbf{X}}, \overline{\beta})$ of complex conjugation on dessins. It is a central involution in Ω. The operations δ, τ and ι generate a subgroup

$$\Omega_1 = \langle \delta, \tau, \iota \rangle \cong D_3 \times C_2 \cong D_6$$

of Ω, preserving the genus of a dessin.

Even more examples of operations

The automorphism $X \leftrightarrow X,\ Y \leftrightarrow Y^{-1}$ of F_2, corresponding to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2\,\mathbb{Z},$$

induces a Petrie operation $\pi \in \Omega$, an involution reversing the rotation of edges around black vertices. This preserves the embedded bipartite graph, but may change the face valencies and the genus of a dessin.

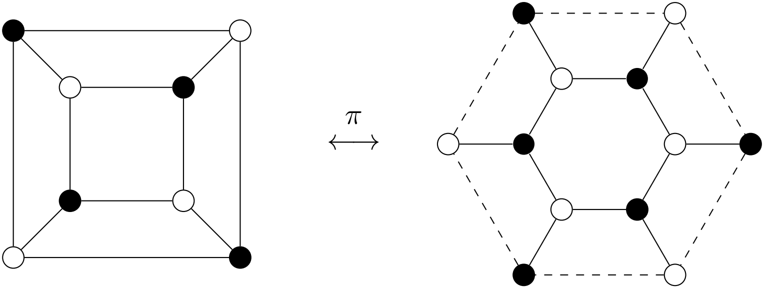


Figure : Sphere and torus embeddings of the cube graph Q_3

Even more examples of operations

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$$\Omega_2 = \langle \delta, \pi \rangle \cong D_4.$$

Free product decomposition

Using a presentation of $\mathrm{GL}_2 \mathbb{Z}$ (Coxeter & Moser, §7.2) one can show that

$$\Omega = \Omega_1 *_{\Omega_0} \Omega_2 \cong D_6 *_{D_2} D_4,$$

the free product of Ω_1 and Ω_2, amalgamating a common subgroup

$$\Omega_0 = \langle \delta, \iota \rangle \cong D_2 \cong C_2 \times C_2.$$

Thus Ω is generated by operations of finite order; there are seven conjugacy classes of these, described by Pinto and J., Discrete Math. 2010.

Invariants of Ω

The operations in Ω preserve

- the monodromy group $G = \mathrm{Mon}\,\mathcal{D}$ of a dessin;
- the automorphism group $A = \mathrm{Aut}\,\mathcal{D}$ of a dessin;
- regularity of a dessin;
- the 'size' $|E|$ of a dessin;
- the cycle structure of the commutator $[x, y]$ acting on E.

However, they do not, in general, preserve the type or the genus of a dessin.

An example

It follows from results of Hall (QJM, 1936) that there are 19 regular dessins $\mathcal{D}$ with automorphism group $A \cong A_5$.

These include the dodecahedron of type $(3, 2, 5)$, the icosahedron of type $(5, 2, 3)$ (both of genus 0), and the great dodecahedron, of type $(5, 2, 5)$ and genus 4 (classified by Breda and J., 2001).

Group-theoretic results of Bernhard and Hanna Neumann (Math. Nachr., 1951) on T-systems show that they form two orbits under Ω of lengths 9 and 10, as $[x, y]$ has order 3 (e.g. the great dodecahedron) or 5 (e.g. the icosahedron and dodecahedron).

Similar ideas

For dessins of type $(p, q, -)$, with $p \neq q$ both fixed, one can replace F_2 with $C_p * C_q$, and $\Omega \cong \mathrm{Out}\, F_2 \cong \mathrm{GL}_2\, \mathbb{Z}$ with

$$\mathrm{Out}\,(C_p * C_q) \cong \mathbb{Z}_p^\times \times \mathbb{Z}_q^\times ,$$

the group of 'Wilson operations' on dessins, raising x and y to primitive powers (Wilson, Pacific J. Math. 1979; Streit, Wolfart and J., PLMS 2010). This generalises the case $(p, 2, -)$, where

$$\mathrm{Out}\,(C_p * C_2) \cong \mathbb{Z}_p^\times$$

acts on p-valent maps (Nedela and Škoviera, PLMS 1997). When $p = q$ one can also include white-black duality δ to give

$$\mathrm{Out}\,(C_p * C_p) \cong \mathbb{Z}_p^\times \wr S_2 \cong (\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times) : S_2.$$

Galois operations

A dessin $\mathcal{D}$ may be identified with a Belyĭ pair $(\mathbf{X}, \beta)$, both defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.

The *absolute Galois group*

$$\mathbb{G} = \mathrm{Gal}\,\overline{\mathbb{Q}} = \mathrm{Aut}\,\overline{\mathbb{Q}}$$

acts on the coefficients of the equations defining $\mathbf{X}$ and β, inducing actions on Belyĭ pairs and hence on dessins.

Examples of orbits of $\mathbb{G}$

Jürgen's talk included an example of a $\mathbb{G}$-orbit of three dessins on the torus. Here is another orbit of length 3, defined over the splitting field of $25t^3 - 12t^2 - 24t - 16$, with $\mathbb{G}$ inducing S_3:

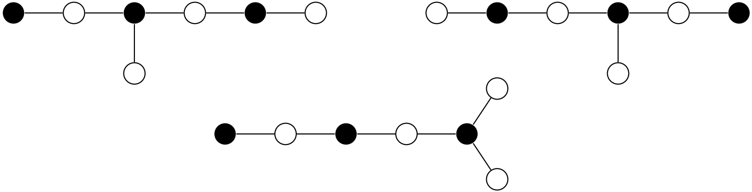

Figure : Three Galois conjugate dessins on the sphere

Galois operations

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The *absolute Galois group*

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acts on the coefficients of the equations defining $\mathbf{X}$ and β, inducing actions on Belyĭ pairs and hence on dessins.

The group $\mathbb{G}$ is very important in algebraic number theory, but it is also very complicated and difficult to work with.

In 1984 Grothendieck suggested studying $\mathbb{G}$ through its action on dessins (and related structures).

Invariants of $\mathbb{G}$

The following properties of a dessin can be defined algebraically, and are therefore invariant under $\mathbb{G}$ (Streit and J., 1997):

- valency distributions of white and black vertices and faces;
- size, type and genus;
- monodromy group and automorphism group.

Faithful action of $\mathbb{G}$

Nevertheless, $\mathbb{G}$ acts faithfully on (isomorphism classes of)

- dessins (Grothendieck);
- dessins of a given genus (Girondo and González-Diez);
- plane trees = maps of genus 0 with one face (Schneps);
- regular dessins (González-Diez and Jaikin-Zapirain);
- regular dessins of a given hyperbolic type (G-D and J-Z).

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Consequence: in principle, one can see 'all' of algebraic number theory by looking at any one of the above classes of dessins.

Practical problem: it is very difficult to give explicit examples of orbits of $\mathbb{G}$ on dessins which reveal much of its structure.

Structure of $\mathbb{G}$

$$\overline{\mathbb{Q}} = \bigcup_{K \in \mathcal{K}} K,$$

where $\mathcal{K}$ is the set of Galois (finite, normal) extensions of $\mathbb{Q}$ in $\mathbb{C}$. For each $K \in \mathcal{K}$ let

$$G_K := \operatorname{Gal} K,$$

a finite group. If $K \geq L$ in $\mathcal{K}$ there is a restriction epimorphism

$$\rho_{K,L} : G_K \to G_L.$$

Then

$$\mathbb{G} = \lim_{\leftarrow} G_K,$$

a profinite group (= projective limit of finite groups). Specifically,

$$\mathbb{G} = \{(g_K) \in \prod_{K \in \mathcal{K}} G_K \mid \rho_{K,L}(g_K) = g_L \text{ whenever } K \geq L\}.$$

Topology on $\mathbb{G}$

$$\mathbb{G} = \{(g_K) \in \prod_{K \in \mathcal{K}} G_K \mid \rho_{K,L}(g_K) = g_L \text{ whenever } K \geq L\}$$

is an uncountable group.

If we put the discrete topology on each G_K then $\prod_{K \in \mathcal{K}} G_K$ is a topological group, compact by Tychonoff's Theorem.

As a closed subgroup, $\mathbb{G}$ is also compact in the induced Krull topology. (Two elements are 'close' if they agree on a large subfield of $\overline{\mathbb{Q}}$.) The topology is that of a Cantor set.

In the Galois correspondence, subfields of $\overline{\mathbb{Q}}$ correspond to closed subgroups of $\mathbb{G}$.

Inverse Galois problem

In the Galois correspondence, subfields of $\overline{\mathbb{Q}}$ correspond to closed subgroups of $\mathbb{G}$.

Hilbert's conjecture that every finite group F is a Galois group over $\mathbb{Q}$ is equivalent to showing that F is a quotient of $\mathbb{G}$ by a closed normal subgroup.

This has been proved for many F (e.g. solvable, symmetric or alternating), but it is still open in general.

Some open problems about dessins

- Find good algorithms for determining the Belyĭ pair $(\mathbf{X}, \beta)$ corresponding to a dessin $\mathcal{D}$ (or at least its moduli field).
- Can one understand Galois orbits without finding explicit models of Belyĭ pairs?
- Find orbits of $\mathbb{G}$ on (regular) dessins on which it induces a (highly) non-abelian group.
- What is the relationship between the groups Ω and $\mathbb{G}$, acting on (regular) dessins? (They do not commute.)

Diolch yn fawr i chi!