The decorated lattice of biased dessins

Jonathan Fine

Open University

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Abstract

Dessins are special types of pairs of permutations of the same set. The absolute Galois group Γ acts on dessins. This is deep and important.

I'm going to explain how to add a little more information to a dessin, called a bias. Then all biased dessins form a lattice, and Γ acts on this lattice.

We can decorate this lattice, to produce *the decorated lattice of biased dessins* $\mathcal{L}$. The absolute Galois group Γ acts on $\mathcal{L}$. This is important.

This leads to interesting questions, the most important of which being: Is the automorphism group of $\mathcal{L}$ equal to Γ? (This is likely to be hard.)

Why do biased dessins appear in the Goulden-Jackson-Rattan study of Stanley's polynomial formula for symmetric group characters?

Also interesting is recent activity in dessins coming from quantum field theory. (Eg A006206: David Broadhurst.) And Γ action on knot invariants.

Permutations, pairs of permutations, and dessins

We situate dessins in the theory of permutations. They're a bit like cycles.

Each permutation α of d objects can be decomposed into cycles, and hence induces a partition p_α of d. Cycles are irreducible.

The partition p_α determines α, up to a relabelling of the objects being permuted. This is called *relabelling equivalence*. Each α has a relabelling normal form, for example (12345)(678)(9) if p_α is $5 + 3 + 1$.

Now let (α, β) be a *pair of permutations* (or *permpair* for short) on the same finite set E, called the *edges* of the pair of permutations.

We can relabel E to put α into normal form, and similarly for β. But only rarely can relabelling put both α and β into normal form at the same time.

A dessin is to a pair of permutations as a cycle is to a permutation. Two edges are in the same dessin if $e_2 = we_1$ for some word w in α and β.

Definition A *dessin* is an irreducible pair of permutations, considered up to relabelling equivalence. A permpair decomposes into dessins.

Product of dessins, and the lattice $\mathcal{L}'$ of biased dessins

We will now define a join operation on 'enhanced dessins' and hence $\mathcal{L}'$.

Let R and S be dessins with edge sets E_R and E_S. The product $E_R \times E_S$ is the edge set for a permpair $R \times S$, via $\alpha_{R\times S}(r,s) = (\alpha_R r, \alpha_S s)$.

Usually $R \times S$ is reducible (e.g. when $R = S$, and R is not trivial) and so is not a dessin. Being able to choose an edge in $R \times S$ would solve this.

Definition A *biased dessin* is a dessin R with a *distinguished edge* e_R.

Definition If R and S are biased dessins, then the *join* $T = R \vee S$ is the component of $R \times S$ that contains $e_T = (e_R, e_S)$. T is a biased dessin.

The projection map $\pi : E_T \to E_R$ clearly (1) sends e_T to e_R, and (2) respects the action of α and β (i.e. $\pi(\alpha t) = \alpha(\pi t)$ etc.)

Definition For any biased dessins T and R we write $T \to R$ if there is such a map between their edges. There is at most one such map (easy).

Theorem The biased dessins are the nodes of a lattice $\mathcal{L}'$. The join $T = R \vee S$ is the least upper bound of R and S (for the $\to$ partial order).

Galois action on biased dessins and invariance of $\mathcal{L}'$

The lattice $\mathcal{L}'$ is important because (1) the absolute Galois group acts on its nodes (biased dessins), and (2) the group action leaves $\mathcal{L}'$ unchanged.

We rely on deep results of Weil, Belyi, Grothendieck and others.

- There is a bijection between dessins and Belyi pairs (maps $M \to \mathbb{P}_1$ from a Riemann surface to $\mathbb{P}_1$ that are unramified away from $0, 1, \infty$).
- Such maps are defined over the algebraic closure $\overline{\mathbb{Q}}$ of the rationals. Thus the *absolute Galois group* Γ (automophisms of $\overline{\mathbb{Q}}$) acts on dessins.

Lemma This bijection and the Γ-action extends to biased dessins and biased Belyi pairs (distinguished point in the fibre above $1/2 \in \mathbb{P}_1$).

Proof The usual proof carries through unchanged to this situation. □

Theorem The lattice structure $\mathcal{L}'$ is Γ-invariant.

Proof For biased dessins T and R there is a map $M_T \to M_R$ (unique if it exists) of covering spaces just in case there is a map $T \to R$. Thus, on biased Belyi pairs the lattice $\mathcal{L}'$ comes from a Γ-invariant property. □

Decorating $\mathcal{L}'$ to obtain $\mathcal{L}$ – nodes

We can decorate $\mathcal{L}'$ in a Γ-invariant way. On this slide we decorate the nodes T, and on the next the maps $T \to R$.

First, we introduce γ, a third permutation that provides additional Γ-invariant information. For Belyi pairs $0, 1, \infty$ all have equal standing.

Belyi 0 and 1 on $\mathbb{P}_1$ corresponds to dessin α and β. Further, Belyi ∞ corresponds to dessin $\gamma = (\alpha\beta)^{-1}$. Therefore, treat α, β and γ similarly.

For each (biased) dessin T we have the permutation α which acts on the edges E_T and hence a partition $p_\alpha T$ on the (number of) edges in T. Adding β and γ gives the *partition triple* $pT = (p_\alpha T, p_\beta T, p_\gamma T)$ of T.

Theorem The partition triple pT of a unbiased dessin T is Γ-invariant.

Proof This is the passport invariant of Lando and Zvonkin. □.

Corollary Attaching to each node T of $\mathcal{L}'$ the partition triple pT provides a Γ-invariant decoration of $\mathcal{L}'$.

Decorating $\mathcal{L}'$ to obtain $\mathcal{L}$ – maps (tricky so just $\mathcal{L}_d$)

Definition T_d is the '*universal at most d-edged biased dessin*'.

T_d is the smallest T such that $T \to R$ for any R with $\leq d$ edges. It is the join of all biased dessin with $\leq d$ edges. (Equivalent to Guillot's H_d?)

Definition Let C be a cycle on T_d. For $T_d \to R$ let C_R be image of C, and $m_C(R)$ the number of edges. Then m_C is the *multiplicity function*.

Definition Set $\mathcal{L}'_d = \{R | T_d \to R\}$. (It is the domain of m_C.)

Definition The *decoration* $\mathcal{L}_d$ of $\mathcal{L}'_d$ is the formal sum (or multiset) of the m_C, over all cycles C on T_d (for α, β and γ separately).

Theorem The decoration of $\mathcal{L}'_d$ is Γ-invariant.

Proof By design, can be done using only local geometry of Belyi pairs. □

Remark We can decorate $\mathcal{L}'$ in a way that restricts to $\mathcal{L}_d$. (Exercise)

Problem Is the restriction map $\text{Aut}(\mathcal{L}_{d+1}) \to \text{Aut}(\mathcal{L}_d)$ surjective?

Problem Is $\mathcal{L}_d$ generated by the biased dessin with $\leq d$ edges?

Decorating $\mathcal{L}'$ to obtain $\mathcal{L}$ – maps (this is tricky)

This slide attaches a *partition map* to each map $T \to R$ in $\mathcal{L}'$. Up to *equivalence*, the system of partition maps is Γ-invariant.

Let p_1 and p_2 be $p_\alpha T$ and $p_\alpha R$ respectively, thought of as non-increasing maps $\mathbb{N}_+ \to \mathbb{N}$. Number the α-cycles of T with initial portion of $\mathbb{N}_+$, etc.

Each cycle of T maps to a cycle of R (because $T \to R$ and α commute). Hence, given a numbering of cycles, we get a map $p_{\alpha,R\to T} : \mathbb{N}_+$ to $\mathbb{N}_+$.

This *partition map*, eventually trivial, is unique up to permutations of $\mathbb{N}_+$ that preserve $p_i : \mathbb{N}_+ \to \mathbb{N}$. This defines *equivalence* of partition maps.

Definition $\mathcal{L}$ is the lattice $\mathcal{L}'$ of biased dessins, decorated with $p_\alpha T$ etc at each node, and the induced $p_{\alpha,T\to R}$ etc at each map $T \to R$.

Theorem The system of partition maps $p_{\alpha,T\to R}$ etc are Γ-invariant (up to renumbering of cycles equivalence).

Proof By design, can be done using only local geometry of Belyi pairs. □

Corollary $\mathcal{L}$ is Γ-invariant (up to equivalence).

Technical summary (and thank you for your attention)

We introduced the *decorated lattice* $\mathcal{L}$ of *biased dessins*.

A dessin is an irreducible pair (α, β) of permutations (on the same finite set of edges). The Cartesian product of two dessins is only a pair of permutations. A *biased dessin* is a dessin with a chosen edge.

If R and S are biased dessins then the Cartesian product $R \times S$ has a distinguished component, denoted by $R \vee S$, which is also a biased dessin. This induces a lattice $\mathcal{L}'$ with nodes the biased dessins.

The permutation α of a dessin T induces a partition $p_\alpha T$ of the (number of) edges of T, and similarly for β and $\gamma = (\alpha\beta)^{-1}$.

Each node T of $\mathcal{L}'$ we decorate with $p_\alpha T, p_\beta T, p_\gamma T$. Each $T \to R$ (i.e. $T = T \vee R$) we decorate with *partition maps* $p_{\alpha, T\to R}$ (tricky).

This defines $\mathcal{L}$. Its automorphism group contains the absolute Galois group (easy, given known hard results). Are the two groups equal?