2-Arc-Transitive regular covers of $K_{n,n} - nK_2$
having the covering transformation group $\mathbb{Z}_p^3$

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Graph Covering: A graph $X$ is called a covering of a graph $Y$ with the projection $p : X \rightarrow Y$ if there is a surjection $p : V(X) \rightarrow V(Y)$ such that $p|_{N(x)} : N(x) \rightarrow N(y)$ is a bijection for any $y \in V(Y)$ and $x \in p^{-1}(y)$.

$X$: Covering graph; $Y$: base graph;

Vertex fibre: $p^{-1}(v), v \in V(Y)$;

Edge fibre: $p^{-1}(e), e \in E(Y)$;
Fibre-preserving automorphism $\sigma \in \text{Aut}(X)$: maps a fibre to a fibre

Covering transformation group $K$:

$$K = \{ \sigma \in \text{Aut}(X) \mid \sigma \text{ fix every fibre setwise.} \}$$

Regular covering: if $K$ acts regularly on each fibre ($X$ is connected)
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2-Arc-Transitive regular covers of $K_{n,n} - nK_2$

$X = Q_3$

$Y = K_4$

$p(a_i) = a, p(b_i) = b$

$p(c_i) = c, p(d_i) = d$

$i = 1, 2$
Fiber: The fiber of an edge or a vertex is its preimage under \( p \);
Fiber preserving automorphism group: An automorphism of \( X \) which maps a fiber to a fiber is said to be fiber-preserving;
Covering transformation group: The group \( K \) of all automorphisms of \( X \) which fix each of the fibers setwise is called the covering transformation group.
Voltage assignment $f$: graph $Y$, finite group $K$ 

a function $f : A(Y) \to K$ s. t. $f_{u,v} = f_{v,u}^{-1}$ for each $(u, v) \in A(Y)$.

Voltage graph: $(Y, f)$

Derived graph $Y \times_f K$: vertex set $V(Y) \times K$, 
arc-set $\{((u, g), (v, f_{u,v}g)) \mid (u, v) \in A(Y), g \in K\}$. 

Shaofei Du 2-Arc-Transitive regular covers of $K_{n,n} - nK_2$
\[ f : \text{A}(Y) \rightarrow K = \{1, 0\} \]

\[ f_{a,b} = 0, \quad f_{a,c} = 1, \quad f_{b,c} = 1 \]

\[ Y = K_3 \]

\[ Y \times_f K \]
Remark:
1. Derived graph $Y \times_f K$ is a covering of $Y$;

2. Derived graph is conn iff voltages on all closed walks generate $K$.

3. Each connected regular covering can be reconstructed by a derived graph.
Lifting: $\alpha \in \text{Aut}(Y)$ lifts to an automorphism $\bar{a} \in \text{Aut}(X)$ if $\alpha p = p\bar{a}$.

Question: Given a graph $Y$, a group $K$ and $H \leq \text{Aut}(Y)$, find all the connected regular coverings $Y \times_f K$ on which $H$ lifts.
For a graph $X$, an $s$-arc of $X$ is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices such that $(v_i, v_{i+1}) \in A(Y)$ and $v_i \neq v_{i+2}$.

$X$ is said to be 2-arc-transitive if $\text{Aut } X$ acts transitively on the set of 2-arcs of $X$. 
$X \rightarrow \text{2-AIG}, \ G = \text{Aut}(X)$

$\text{no } g \in \text{Aut}X, \ (a, b, c_1) \rightarrow (a, b, c_2)$
1. Quasipri.

2. Bipartite

3. Cover
Praeger’s Reduction Theorem

Theorem

Every finite connected 2-arc-transitive graph is one of the following:

(1) Quasiprimitive Type: every non-trivial normal subgroup of $\text{Aut} X$ acts transitively on $V(X)$,

(1) Bipartite Type: every non-trivial normal subgroup of $\text{Aut} X$ has at most two orbits on $V(X)$ and at least one of normal subgroups of $\text{Aut} X$ has exactly two orbits on $V(X)$.

(3) Covering Type: There exists a normal subgroup of $\text{Aut} X$ which has at least three orbits on $V(X) \rightarrow$ regular covers of graphs in (1) and (2).

Every primitive group must be quasiprimitive, but a quasiprimitive group is not necessarily primitive.
Example:

\[ G = \text{PSL}(2, 7) \]

acts on the edge set \( E \) of point-line incident graph of \( \text{PG}(2, 2) \), where \( |E| = 21 \)

for any edge \( e \), we have \( G_e = D_8 \leq S_4 \leq G \) and so \( G \) is an imprimitive group on \( E \).


Locally primitive graphs...and so on
A reduction theorem for this case was given by Praeger (1993).

In sense of group theory, it induces two directions:

(1) Study Quasiprimitive type

→ to study finite $G$, which is a simple group, almost simple group, Quasisimple group, primitive group, Quasiprimitive and so on, and to study the suborbit structures of the related permutation representations:

$H = G_\alpha$ for $\alpha \in V$

$[G : H] =$ the set of right coset of $H$ in $G$

the neighbor of $\alpha = HgH$

the induced action of $H$ on $HgH$ is a 2-transitive group.
Bipartite type:

\[ A = \text{Aut}(X) \]

\( G \) is the subgroup \( A \) fixing two biparts setwise and \( G \) be one of group as in (1)

\( H = G_{\alpha} \text{ and } R = G_{\beta} \text{ for } \{\alpha, \beta\} \text{ is an edge} \)

\([G : H], [G : R] \text{ the set of right coset of } H \text{ and } R \text{ in } G, \text{ resp.} \)

the neighbor of \( \alpha = RgH \)

the induced action of \( H \) on \( RgH \) is a 2-transitive group.
(2) Study regular covers of Quasiprimitive or Bipartite type to study the group extensions of the above groups, in many cases, it is related central extension theory as well as Schur Multiplier theory, (ordinary and in most cases, modular) representations of almost simple groups and so on.
One of our long term topics is to classify covers whose fibre preserving group acts 2-arc-transitively, by given base graphs (2-ATG of either Quasiprimitive type or Bipartite type)
given covering transformation groups (\(\mathbb{Z}_p^n\), abelian groups and nonabelian groups)
3. Some Classifications and Methods

1. Present some classifications of 2-arc-transitive regular covers with given base graphs and given covering transformation groups.

2. Show our general methods for classifying the 2-arc-transitive covers and for constructing voltage graphs.
Problem: Classify regular covers of complete graphs having the covering transformation group $K = \mathbb{Z}_p^k$ and whose fibre-preserving group acts 2-arc-transitively.

Motivation: If $1 \neq H \vartriangleleft \vartriangleleft K$, then $X \to X_1$ and $X_1 \to K_n$

$X = X_1 \times_{f_2} (H)$ and $X_1 = K_n \times_{f_1} (K/H)$

First need to determine the minimal covers, that is $K$ is a characteristically simple group. Therefore, we choose $K = \mathbb{Z}_p^k$, an abelian characteristically simple group.

**Theorem**

If $Y = K_n$ and $K$ is cyclic, then
(i) $K \cong \mathbb{Z}_2$ and $X = K_{n,n} - nK_2$;
(ii) $K \cong \mathbb{Z}_4$ and $X \cong X_1(4,q)$, where $q = n - 1$;

If $Y = K_n$ and $K = \mathbb{Z}_p^2$, then
$K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $X \cong X_2(4,q)$, where $q = n - 1$. 
Definitions for $X_1(4, q)$ and $X_2(4, q)$:

Let $Y = K_n$ with $V(Y) = \text{PG}(1, q) = F(q) \cup \{\infty\}$ where $F(q)^* = \langle \theta \rangle$

$X_1(4, q) = Y \times_f \mathbb{Z}_4$, where $q \equiv 3 (\text{mod } 4)$ and $q \geq 3$:

$$f(x, y) = \begin{cases} 
0, & \infty \in \{x, y\} \\
1, & y - x \in \langle \theta^2 \rangle \\
3, & y - x \in \langle \theta^2 \rangle \theta 
\end{cases}$$
$X_2(4, q) = Y \times_f \mathbb{Z}_2^2$, where $q \equiv 1 (mod \ 4)$ and $q \geq 5$:

$$f(x, y) = \begin{cases} 
(0, 0), & \infty \in \{x, y\} \\
(1, 0), & y - x \in \langle \theta^2 \rangle \\
(0, 1), & y - x \in \langle \theta^2 \rangle \theta
\end{cases}$$

Remark: for all both covers, $\text{PGL}(2, q)$ lifts and so $X$ is a 2-ATG.

**Theorem**

If \( Y = K_n \) and \( K = \mathbb{Z}_p^3 \), then we have

(i) \( X_1(p) = K_4 \times_f \mathbb{Z}_p^3 \),

(ii) \( X_2(p) = K_5 \times_f \mathbb{Z}_p^3 \) for \( p = 5 \) or \( p \equiv \pm 1(\text{mod } 10) \),

(iii) \( X_3(p) = K_{1+p} \times_f \mathbb{Z}_p^3 \) for \( p \geq 5 \),

(iii) \( X_4(3) = K_8 \times_f \mathbb{Z}_2^3 \).
$k = 4$

Open
For a Cayley graph, its automorphism group contains a vertex-regular subgroup.

Cayley graphs of cyclic and dihedral groups are called \textit{Circulant} and \textit{Dihedrants}, respectively.


The proof is combinatorial and is independent on CFSG.

S.F. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, J. Combin. Theory, B, 98(6), (2008), 1349-1372
**Theorem**

Let $n \geq 3$ and let $X$ be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$. Then one of the following occurs:

3mm

(i) base graph: $C_{2n}$, $n$ a prime; $K_{2n}$; $K_{n,n}$; $B(H_{11})$ or $B'(H_{11})$; $B(PG(d, q))$ or $B'(PG(d, q))$, or

(ii) $K_{n,n} - nK_2$ or $K_{q+1}^{2d}$. 

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\[ Y = K_{q+1,q+1} - (q + 1)K_2, \]

\[ V(Y) = \{i, j' \mid i, j \in \text{PG}(1, q)\} \]

\[ E(Y) = \{i, j' \mid i \neq j, i, j \in \text{PG}(1, q)\}. \]

\[ K_{q+1}^{2d} = (K_{q+1,q+1} - (q + 1)K_2) \times_f \mathbb{Z}_d, \text{ where} \]

\[ f_{\infty,i} = f_{\infty,j'} = 0 \text{ for } i, j \neq \infty; \]

\[ f_{i,j'} = h \text{ if } j - i = \theta^h, \text{ for } i, j \neq \infty, \]

where \( F_q^* = \langle \theta \rangle \), \( d \mid (q - 1) \) and \( d \geq 2. \)
3.3. 2-Arc-Transitive Metacyclic Covers of Complete Graphs

Theorem

Let $X$ be a connected regular cover of the complete graph $K_n$ ($n \geq 4$) whose covering transformation group $K$ is nontrivial metacyclic and whose fibre-preserving automorphism group acts 2-arc-transitively on $X$. Then $X$ is isomorphic to one of covers:

1. $K_{n,n} - nK_2$;
2. $n = 4$, $AT_D(4, 6)$ with $K \cong D_6$;
3. $n = 4$, $AT_Q(4, 12)$ with $K \cong Q_{12}$;
4. $n = 5$, $AT_D(5, 6)$ with $K \cong D_6$;
5. $n = 1 + q \geq 4$, $AT_Q(1 + q, 2d)$ with $K \cong Q_{2d}$, where $d \mid q - 1$ and $d \nmid \frac{1}{2}(q - 1)$;
6. $n = 1 + q \geq 6$, $AT_D(1 + q, 2d)$ with $K \cong D_{2d}$, where $d \mid \frac{1}{2}(q - 1)$ and $d \geq 2$.

where $Q_{2d}$ is the generalized quaternion group of order $2d$ and $D_{2d}$ is the dihedral group of order $2d$. Note that $Q_4 \cong \mathbb{Z}_4$ and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. 
3.4. Cyclic regular covers of $K_{n,n} - nK_1$

Let $X$ be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ ($n \geq 4$) with a nontrivial cyclic covering transformation group of order $d$, whose fibre-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:

1. $n = 4$ and $X \cong K_4^4$ for $d = 2$; $X_1(3)$ for $d = 3$; or $X(6)$ for $d = 6$;
2. $n = 5$ and $X \cong X_2(3)$ for $d = 3$;
3. $n = q + 1 \geq 6$ and $X \cong K_{q+1}^{2d}$, where $d \mid q - 1$ and $d \geq 2$. 

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2-Arc-Transitive regular covers of $K_{n,n} - nK_2$
4. 2-Arc-Transitive regular covers of $K_{n,n} - nK_2$ having the covering transformation group $\mathbb{Z}_p^3$

Theorem

Let $X$ be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ ($n \geq 3$) with a covering transformation group $K$ isomorphic to $\mathbb{Z}_p^3$, whose fiber-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:

1. $n = 4$ and $X \cong X_1(4, p)$;
2. $n = 5$ and $X \cong X_{21}(5, p)$ for $p \equiv \pm 1 \pmod{10}$ or $X_{22}(5, 5)$ for $p = 5$;
3. $n = p + 1 \geq 6$ and $X \cong X_{31}(p + 1, p)$ for $p \geq 5$, or $X_{32}(6, 5)$ for $p = 5$;
4. $n = 8$ and $X \cong X_4(8, 2)$ for $p = 2$. 
$Y := K_{n,n} - nK_2$ and $K \cong \mathbb{Z}_p^3$, voltage assignment $f$, where $V(Y) = \{i, i' \mid 1 \leq i \leq n\}$, $E(Y) = \{ij' \mid i \neq j, i, j' \in V(Y)\}$ and $K = V^+(3, p)$.

(1) $n = 4$ and $X_1(4, p) = Y \times_f K$, where

\[
\begin{align*}
    f_{12'} &= f_{13'} = f_{14'} = f_{24'} = f_{21'} = f_{31'} = f_{41'} = (0, 0, 0), \\
    f_{23'} &= (1, 0, 0), \\
    f_{42'} &= (0, 1, 0), \\
    f_{34'} &= (0, 0, 1), \\
    f_{43'} &= (0, 1, -1), \\
    f_{32'} &= (-1, 1, 0).
\end{align*}
\]
(2) \( n = 5, \ p = \pm 1(\text{mod } 10) \) and \( X_{21}(5, p) = Y \times f K \), where

\[
\begin{align*}
  f_{1,2'} &= (0, 2t, 1 - 2t), \\
  f_{1,4'} &= (1 - 2t, 0, 2t), \\
  f_{2,3'} &= (1 - 2t, 0, -2t), \\
  f_{2,5'} &= (-1, 1, 1), \\
  f_{3,5'} &= (1, 1, -1), \\
  f_{i,j'} &= f_{i',j}, \quad i, j \in \{1, 2, 3, 4, 5\}, \\
  f_{1,3'} &= (2t, 1 - 2t, 0), \\
  f_{1,5'} &= (-1, -1, -1), \\
  f_{2,4'} &= (2t, 2t - 1, 0), \\
  f_{3,4'} &= (0, -2t, 1 - 2t), \\
  f_{4,5'} &= (1, -1, 1), \\
\end{align*}
\]

where \( t = \frac{1 + \sqrt{5}}{4} \in \mathbb{F}_p^* \).

\( n = p = 5 \) and \( X_{22}(5, 5) = Y \times f K \), where

\[
\begin{align*}
  f_{1,2'} &= (0, -1, 0), \\
  f_{1,4'} &= (2, 3, -1), \\
  f_{2,3'} &= (0, -1, 3), \\
  f_{2,5'} &= (2, 2, -1), \\
  f_{3,5'} &= (3, 1, 2), \\
  f_{i,j'} &= f_{i',j}, \quad i, j \in \{1, 2, 3, 4, 5\}. \\
\end{align*}
\]
(3) **Label** \( V(Y) = \{i, j' \mid i, j \in PG(1, p)\} \) and \( E(Y) = \{ij' \mid i, j' \in V(Y), i \neq j\} \).

\( n = 1 + p, \ p \geq 5 \) and \( X_{31}(p + 1, p) = Y \times_f K \), where \( f_{\infty,i'} = f_{\infty,i} = (0, 1, 2i) \), and \( f_{i,j'} = f_{i',j} = \left( \frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j} \right) \) for all \( i \neq j \) in \( \mathbb{F}_p \).

\( n = 6, \ p = 5 \) and \( X_{32}(6, 5) = Y \times_f K \), where \( f_{\infty,i'} = f_{\infty,i} = (-i, -i^2, i^3) \), \( f_{i,j'} = (0, \pm 2, \pm 2(i + j)) \) for \( (i - j)^2 = \mp 1 \), where \( i, j \in \mathbb{F}_5 \).
Let $\Omega = \text{PG}(2, 2)$;

$V = V(\Omega)$: the characteristic functions $\chi(\Delta)$, $\Delta \in P(\Omega)$;

$V$ is a 7-dimensional $\text{PSL}(2, 7)$-module by natural action;

$V_1$: the subspaces of $V$ generated by
\[
\{i, j, i + j \mid i, j \in \Omega, i \neq j\}
\]

$V_2$: the subspaces of $V$ generated by
\[
\{i, j, k, i + j, i + k, j + k, i + j + k \mid i, j, k \in \Omega\}.
\]
Let $Y = K_{8,8} - 8K_2$,
$V(Y) = \{i, j' \mid i, j \in V(3, 2)\}$
$E(Y) = \{ij' \mid i, j' \in V(Y), i \neq j\}$
$K = (V_1/V_2, +)$
$n = 8, p = 2$ and $X_4(8, 2) = Y \times f K$, where

$f_{0,j'} = 0 := V_2$ and $f_{i,j'} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2$ for all $i \neq j$ in $\Omega$. 
base graph $Y = K_{n,n} - nK_2$,

covering graph $X$,

covering transformation group $K = Z_3^p$

$A$: 2-arc-transitive subgroup of $\text{Aut}(Y)$

$G$: subgroup of $A$ fixing two biparts setwise,

$G$ is 3-transitive on both biparts
$G$ is the following:

(1) $\text{soc}(G)$ is 4-transitive;
(2) $\text{soc}(G) = M_{22}$ or $A_5$, 3-transitive but not 4-transitive;
(3) $\text{PSL}(2, q) \leq G \leq \text{PGL}(2, q)$;
(4) $G = \text{AGL}(m, 2)$ with $m \geq 3$ or $\mathbb{Z}_2^4 \rtimes A_7$. 
Main problem is to determine the group extension

\[ 1 \to K \to \tilde{G} \to G \]

\[ 1 \to K \to \tilde{A} \to A \]
The key point in graph (regular-) covering theory is the lifting problem of automorphisms in the base graphs, which is related to several branches.
5.1 Methods from topological graph theory

Lifting criterion for any covering group $K$:

Theorem

Let $X = Y \times_f K$ be a regular covering. Then $\alpha \in \text{Aut}(Y)$ lifts if and only if, for each closed walk $W$ in $Y$, we have $f_W \alpha = 1$ iff $f_W = 1$.

Lifting criterion for abelian covering group $K$:

**Theorem**

(Du, Kwak, Marusic, Waller, Xu) Let $X = Y \times_f K$ be a connected regular cover of a graph $Y$, where $K$ is abelian. If $\alpha \in \text{Aut } Y$ is an automorphism one of whose liftings $\tilde{\alpha}$ centralizes $K$, then $f_{W^\alpha} = f_W$ for any closed $W$ of $Y$. 
Linear criteria for liftings of automorphisms for elementary abelian covering group $K = \mathbb{Z}_p^n$


Linear criteria for liftings of automorphisms for abelian covering group


Example:
5.2 Methods from group theory

Coset Graphs:

**Definition**

- \( G \) group; \( H \leq G \), core-free; \( D = HdH \) double coset
- Coset graph \( X = X(G; H, D) \):
  - \( V = [G : H] \): right cosets of \( G \) relative \( H \)
  - \( E = \{(Hd, Hdg) \mid d \in D, g \in G\} \)

**Lemma**

1. \( G \) acts transitively on \( X \);
2. Every \( G \)-arc-transitive graph is isomorphic to a coset graph.
3. \( X \) is connected iff \( G = \langle D \rangle \);
4. \( X \) is undirected iff \( D = D^{-1} \).
Bicoset Graph:

**Definition**

Let $G$ be a group, $L, R \leq G$, $D = RdL$.

The bicoset graph $X = (G, L, R; D)$ is defined as:

- $V(X) = [G : L] \cup [G : R]$
- $E(X) = \{ \{Lg, Rdg\} \mid g \in G, d \in D\}$.

**Lemma**

(i) $G$ acts edge-transitively;

(iii) $X$ is connected if and only if $G$ is generated by elements of $D^{-1}D$;

(iv) Every $G$-edge-transitive graph is isomorphic to a Bicoset graph.
Let $X = Y \times_f K$ and let $G \leq \text{Aut}(Y)$. Suppose that $G$ lifts to $A$. Then $A/K \cong G$.

**Step 1:** Determine the group $A$ by the group theoretical tools (group extension, representation theory and so on);

**Step 2:** Determine the permutation representations of $A$ relative to all the possible subgroups $H$ (point stabilizer);

**Step 3:** Determine the corresponding suborbit structure of the above representations so that obtain the coset graphs;

**Step 4:** Find the voltage assignment from these coset graphs.
Example 1

Let $Y = K_{1+p}$ where $V(Y) = PG(1, p) = GF(p) \cup \{\infty\}$ and let $K = (V(3, p), +)$. Find all the regular coverings $X = Y \times_f K$ such that $PGL(2, p) \leq Aut(Y)$ lifts.

Solution: (1) Define $X(p) =: K_{1+p} \times_f Z_p^3$ as follows:

$f_{\infty,j} = (0, 1, 2j),

f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $GF(p)$.

(2) $X'(5) = K_6 \times_f Z_p^3$ as follows:

....
Proof for Example 1

\[ \frac{A}{K} \cong \text{PGL}(2, p) \quad \text{for } p \geq 5 \quad \text{and} \quad n = 1 + p. \]

Take a fibre \( F \) and a vertex \( v \in F \). Then \( A_F = A_vK \).

Since \( (|A : A_F|, |K|) = (1 + p, p^3) = 1 \) and \( K \) is an abelian normal subgroup of \( A \), we know that \( K \) has a complement in \( A \) which is isomorphic to \( \text{PGL}(2, p) \), that is

\[ A \cong \mathbb{Z}_p^3 \rtimes \text{PGL}(2, p) \]

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Modular $p$- Representations of 2-dimensional linear groups:


2. R. Burkhardt, Die Zerlegungsmatrizen de Gruppen $PSL(2, p^f)$, J. Algebra of Algebra, 40(1976), 75-96

$SL(2, p)$ has $p$ irreducible modular $p$- Representations

$PSL(2, p)$ has $\frac{p+1}{2}$ irreducible modular $p$- Representations

with degrees $1, 3, 5, \cdots, p$
Degree 3:

\[ V_3 = \langle x^i y^j \mid i + j = 2 \rangle \text{ homogeneous space over } F_p \]

\[ g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

Define \( G = \text{PSL}(2, p) \)-module \( V_3 \) extended by

\[ g(x^i y^j) = (a_{11}x + a_{12}y)^i(a_{21} + a_{22}y)^j \]

Let \( G = \text{PGL}(2, p) \). Define two \( G \)-modules \( V_3 \) extended by

\[ g(x^i y^j) = \det(g)^{-1}(a_{11}x + a_{12}y)^i(a_{21} + a_{22}y)^j \]

and

\[ g(x^i y^j) = \det(g)^{\frac{p-1}{2}} (a_{11}x + a_{12}y)^i(a_{21} + a_{22}y)^j \]
Take a base in $V_3$, we get two homomorphisms $\phi$ of $\text{PGL}(2, p)$ into $\text{GL}(3, p)$

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad - bc)^{-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad + bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$  

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad - bc)^{p-1/2} - 1 \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad + bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$  

Note: The first case will give the covers $X(p)$ the second will gives the covers $X'(5)$. 
Step 2: Determination of conjugacy class of point stabilizers

Take a subgroup $H_1 = \langle t_1 \rangle \rtimes \langle a_1 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_{p-1}$ of $\text{PGL}(2, p)$, where

$$t_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

for a generator $\theta$ of $\text{GF}(p)^*$. Let $PG(1, p) = \{\infty, 0, 1, \ldots, p - 1\}$ be the projective line over $\text{GF}(p)$, where we identify $\langle (0, 1) \rangle$ and $\langle (1, \ell) \rangle$ with $\infty$ and $\ell$, respectively. Then, $H_1$ fixes $\infty \in PG(1, p)$ and $t_1^i$ maps $\ell$ into $\ell + i$. Furthermore, we have $H := \phi(H_1) = \langle t \rangle \rtimes \langle a \rangle$, where $t = \phi(t_1)$ and $a = \phi(a_1)$, and for any $i$,

$$t^i = \phi(t_1^i) = \begin{pmatrix} 1 & 2i & 2i^2 \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a^i = \phi(a_1^i) = \begin{pmatrix} \theta^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^{-i} \end{pmatrix}.$$
Lemma

Let $M = K \rtimes H$. Then, $M$ has only one conjugate class of subgroups $L$ satisfying $\langle a \rangle \leq L \cong H$ and $L \cap K = 1$. 
Proof  Note that
\[ |M| = |K \rtimes H| = |(K \rtimes \langle t \rangle) \rtimes \langle a \rangle| = p^4(p - 1). \]
Let \( P = K \rtimes \langle t \rangle. \) Then, \( P \) is a \( p \)-group of order \( p^4. \) Since \( p \geq 5 \) by assumption, \( P \) is a regular \( p \)-group (for the definition of regular \( p \)-groups.

Since \( \Phi(P) \leq K \) and the order of \( t \) is \( p, \) \( P \) has exponent \( p. \)

Clearly, \( M \) has only one conjugacy class of subgroups isomorphic to \( \langle a \rangle. \) Assume that \( L \) is a subgroup of \( M \) such that \( \langle a \rangle \leq L \cong H \) and \( L \cap K = 1. \) Then, we may assume that \( L = \langle kt \rangle \rtimes \langle a \rangle \) for some \( k = (x, y, z) \in K. \) Suppose that
\[ (kt)^a = (kt)^i. \]
Then, we have \( (kt)^a = k^a t^a = (\theta x, y, \theta^{-1}z) t^{\theta^{-1}} \)
\[(kt)^i = (kkt^{-1}k^{t-2} \cdots k^{t-i+1})t^i = ((x, y, z) + (x, -2x + y, 2x - 2y + z) + \cdots
+ (x, -2(i - 1)x + y, 2(i - 1)^2x - 2(i - 1)y + z))t^i
= (ix, -(i - 1)ix + iy, \frac{(i - 1)i(2i - 1)}{3}x - (i - 1)iy + iz)t^i.\]

Thus, we get \(i = \theta^{-1}\) and

\[(\theta x, y, \theta^{-1}z) = (ix, -(i - 1)ix + iy, \frac{(i - 1)i(2i - 1)}{3}x - (i - 1)iy + iz).\]

From these two equations, we have \(\theta x = ix = \theta^{-1}x\) and so \(\theta^2x = x\). Since \(p \geq 5\), we get \(\theta^2 \neq 1\), and so \(x = 0\) and \(y = 0\) by the second equation again. Hence, \(k = (0, 0, z)\) for any \(z \in GF(p)\), that means \(k\) has \(p\) possibilities. For each \(k\), we get an \(L = \langle kt \rangle \rtimes \langle a \rangle\); in particular, \(L = H\) when \(z = 0\). Furthermore, these \(p\) subgroups are conjugate in \(M\).
In fact, for any \( k = (0, 0, z) \), by taking \( k' = (0, \frac{z}{2}, 0) \), we have

\[
(kt)^{k'} = k(k')^{-1}tk' = k(k')^{-1}(k')^{t^{-1}}t \\
= ((0, 0, z) - (0, \frac{z}{2}, 0) + (0, \frac{z}{2}, -z))t = (0, 0, 0)t = t
\]

\[
a^{k'} = k'^{-1}ak' = k'^{-1}(k')^{a^{-1}}a = \left((0, -\frac{z}{2}, 0) + (0, \frac{z}{2}, 0)\right)a = a,
\]

which forces \( L^{k'} = H \), completing the proof.
Step 3: Determination of suborbits of $A$ relative to $H$

**Lemma**

Let $[A : H]$ be the set of right cosets of $H$ in $A$. Then, in its right multiplication action on $[A : H]$, $A$ has $p - 1$ suborbits of length $p$ not contained in $[M : H]$, which correspond to the $p - 1$ double cosets $Hg(0, y, 0)H$ for any $y \in GF(p)^*$ and $g = \phi(g_1)$, where

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Proof  Suppose that the double coset $D$ corresponds to a suborbit of $A$ of length $p$ relative to $H$ not contained in $[M : H]$. Since $H$ has only one conjugacy class of subgroups of order $p - 1$, $a$ must fix a point in this suborbit.

Noting that $T$ is 2-transitive on $[T : H]$, we may choose $D = HgkH$ such that $Hgk = Hgka$, in other words, $Hg = Hg^{-1}k^a$, which forces that $Hg = Hga^{-1}$ and $k^a = k$.

Hence, we may fix $g = \phi(g_1)$. Assume $k = (x, y, z)$. From $(\theta x, y, \theta^{-1} z) = k^a = k = (x, y, z)$, we have $x = z = 0$ as $\theta \neq \pm 1$, and so $k = (0, y, 0)$, where $y \neq 0$. Therefore, we get $p - 1$ choices for $k$ and so for $D$. □
Step 4: Determination of Coset graphs

Now, $M = K \rtimes H = AF$ for a fibre $F$. For any $u \in F$, we have $M_u \cong H$ and $M_u \cap K = 1$. Since $M$ has only one conjugacy class of subgroups isomorphic to $\langle a \rangle$, there exists a vertex $v \in F$ such that $\langle a \rangle \leq M_v$. By Lemma ??, $M_v$ is conjugate to $H$ in $M$. It follows that $H$ fixes a vertex in $F$. Therefore, $X$ is isomorphic to one of $X(A, H, D)$, where $D = Hg(0, y, 0)H$ is as in Lemma 0.6. Moreover, it is easy to see that the $p - 1$ graphs corresponding to the $p - 1$ choices for $D$ are isomorphic to each other, by changing the basis of $V(3, p)$.

Now, we may choose $k = (0, 1, 0)$. Note that

$$g = \phi(g_1) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$ 

Since $(gk)^2 = 1$, we get $D = HgkH = H(gk)^{-1}H = D^{-1}$. So, $X(A, H, D)$ is an undirected graph. Clearly, $A$ acts 2-arc-transitively on $X(A, H, D)$, because $T$ is 3-transitive on $V(K_n)$. 

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Step 5: Determination of the voltage assignment

Lemma

\[ X(A, H, D) \cong X(p), \text{ and its group of fibre-preserving automorphisms acts 2-arc-transitively.} \]

Proof  Considering the action of \( \text{PGL}(2, p) \) on \( \text{PG}(1, p) \), one can easily check that for \( \ell \in GF(p)^* \) both \( g_1 t_1^\ell g_1 t_1^i \) and \( g_1 t_1^{i-\ell^{-1}} \) map \( \infty \) to \( i - \ell^{-1} \), respectively. Since \( (\text{PGL}(2, p))_{\infty} = H_1 \), we have that for any \( i \in GF(p) \), \( H_1 g_1 t_1^\ell g_1 t_1^i = H_1 g_1 t_1^{i-\ell^{-1}} \) and so under the homomorphism \( \phi \) mentioned before \( Hgt_1^\ell gt_1^i = Hgt_1^{i-\ell^{-1}} \). In addition, \( (Hg)gt_1^i = H \).

By the arguments before the lemma, we know that in the coset graph \( X(A, H, D) \), \( H \) is adjacent to \( Hgkt_1^\ell \) for any \( \ell \in GF(p) \). Hence, for any \( i \in GF(p) \), \( Hgt_1^i \) is adjacent to \( Hgkt_1^\ell gt_1^i = Hgt_1^\ell gt_1^i k(t_1^\ell gt_1^i) \) for any \( \ell \in GF(p) \).
If \( \ell = 0 \), then

\[
Hgt^\ell gt^i k(t^\ell gt^i) = H(0, 1, 0)gt^i = H(0, -1, -2i).
\]

Hence, \( Hgt^i \) is adjacent to \( H(0, -1, -2i) \) for any \( i \in GF(p) \), or equivalently, \( H \) is adjacent to \( Hgt^j(0, 1, 2j) \) for any \( j \in GF(p) \).

Assume \( \ell \in GF(p)^* \) and let \( i - \ell^{-1} = j \). Then,

\[
Hgt^\ell gt^i k(t^\ell gt^i) = Hgt^{i-\ell^{-1}}(0, 1, 0) t^\ell gt^i
= Hgt^{i-\ell^{-1}}(\ell, 2i\ell - 1, 2i^2\ell - 2i) = Hgt^j \left( \frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j} \right)
\]

Hence, \( Hgt^i \) is adjacent to \( Hgt^j \left( \frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j} \right) \) for any \( i \neq j \in GF(p) \).

Considering the action of \( PGL(2, p) \) on \( PG(1, p) \), we may define a bijection from \([PGL(2, p) : H_1]\) to \( PG(1, p) \) by sending \( H_1 \) to \( \infty \) and \( H_1 g_1 t^i \) to \( i \). Accordingly, we may define a bijection from \([T : H]\) to \( PG(1, p) \) by sending \( H \) to \( \infty \) and \( Hgt^i \) to \( i \).
Finally, we may define a map $\sigma$ from $V(X(A, H, D))$ to $V(X(p)) = PG(1, p) \times K$ by sending $Hk$ to $(\infty, k)$ and $Hgt^i k$ to $(i, k)$. In viewing the above arguments and the definition of $X(p)$, we find that $\sigma$ is an isomorphism from $X(A, H, D)$ to $X(p)$. Moreover, since $A$ acts 2-arc-transitively on $X(A, H, D)$, it follows that for the graph $X(p)$, its group of fibre-preserving automorphisms acts 2-arc-transitively.
Step 6: Generalize to $X(p)$ to $X(q)$

**Lemma**

For each cover in $X(q)$, the group of fibre-preserving automorphisms acts 2-arc-transitively.

**Proof** Recall that $V(K_{1+q})$ is identified with the projective line $PG(1, q) = GF(q) \cup \{\infty\}$. We will adopt the usual computations between $\infty$ and the elements in $GF(q)$, that is, $\infty + i = \infty$ for $i \in GF(q)$; $\infty i = \infty$ for $i \in GF(q)^*$; and $\frac{\infty}{\infty} = 1$.

Let $K$ be the corresponding additive group of $V(3, q)$. Then, $X(q) = K_{1+q} \times_f K$ is defined by $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $PG(1, q)$.

To prove the lemma, it suffices to show that $PGL(2, q)$ lifts.

For a computation, we identify the element $\infty$ and any $i \in GF(q)$ in $PG(1, q)$ with $\langle(1, 0)\rangle$ and $\langle(i, 1)\rangle$ respectively.

For a matrix $g$ in $GL(2, q)$, we denote by $\overline{g}$ the image of $g$ in $PGL(2, p^\ell)$ under the natural homomorphism.
Then, the action of $\bar{g} \in \text{PGL}(2, p^\ell)$ on $\infty$ and any $i \in \text{PG}(1, p^\ell)$ can be written respectively as follows:

$$\infty \bar{g} := \langle g(1, 0) \rangle \quad \text{and} \quad i \bar{g} := \langle g(i, 1) \rangle.$$ 

Let

$$g_1 = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad g_4 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

where $x$ is a primitive element in $GF(q)$. Then, all of these elements generate $\text{PGL}(2, q)$. In addition, it is easy to check that

$$i \bar{g}_1 = ix^2, \quad i \bar{g}_2 = i + 1, \quad i \bar{g}_3 = \frac{i}{i + 1}, \quad i \bar{g}_4 = ix,$$

where $i \in \text{PG}(1, q)$. In what follows, we show that for $1 \leq k \leq 4$, $\bar{g}_k$ lifts.

Let $W$ be a closed walk in $Y$ with $f_W = 0$, and for any arc $(i, j) \in A(Y)$, let $\ell_{i,j}$ has the same notation as above.
Now, we get

\[ f_W = \sum_{(i,j) \in A(Y)} \ell_{i,j} f_{i,j} = \sum_{(i,j) \in A(Y)} \ell_{i,j} \left( \frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j} \right) = 0. \]

Therefore, we have

\[ \sum_{(i,j) \in A(Y)} \frac{\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j) \in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j) \in A(Y)} \frac{2ij\ell_{i,j}}{i-j} = 0. \]

Also, we have

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Now, we get

\[
f_{W^g_1} = \sum_{(i,j) \in A(Y)} \ell_{i,j} f_{ig_1 jg_1} = \sum_{(i,j) \in A(Y)} \ell_{i,j} f_{ix^2,ix^2} = \sum_{(i,j) \in A(Y)} \ell_{i,j} \left( \frac{1}{i x^2 - j x^2}, \frac{i x^2 + j x^2}{i x^2 - j x^2}, \frac{2 i x^2 j x^2}{i x^2 - j x^2} \right) = x^{-2} \sum_{(i,j) \in A(Y)} \ell_{i,j} \cdot \sum_{(i,j) \in A(Y)} \frac{(i+j) \ell_{i,j}}{i-j}, x^2 \sum_{(i,j) \in A(Y)} \frac{2 i j \ell_{i,j}}{i-j} = 0.
\]

Similarly, we get that \( f_{W^{g_k}} = 0 \), for \( k = 2, 3 \) and 4. By Proposition 8, \( g_k \) lifts, and so \( \text{PGL}(2, q) \) lifts.
Thank You Very Much!