Platonic solids generate their four-dimensional analogues

Pierre-Philippe Dechant

Mathematics Department, Durham University

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1 Introduction
   - Coxeter groups and root systems
   - Clifford algebras
   - ‘Platonic’ Solids

2 Combining Coxeter and Clifford
   - The Induction Theorem – from 3D to 4D
   - Automorphism Groups
   - Trinities and McKay correspondence
**Root systems – \( A_2 \)**

A root system \( \Phi \) is a set of vectors \( \alpha \) such that:

1. \( \Phi \cap \mathbb{R}\alpha = \{ -\alpha, \alpha \} \quad \forall \alpha \in \Phi \)
2. \( s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi \)

Simple roots: express every element of \( \Phi \) via a \( \mathbb{Z} \)-linear combination (with coefficients of the same sign).

Platonic solids generate their four-dimensional analogues.
A Coxeter group is a group generated by some involutive generators $s_i, s_j \in S$ subject to relations of the form $(s_is_j)^{m_{ij}} = 1$ with $m_{ij} = m_{ji} \geq 2$ for $i \neq j$.

The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space $\mathcal{E}$. In particular, let $(\cdot|\cdot)$ denote the inner product in $\mathcal{E}$, and $v, \alpha \in \mathcal{E}$.

The generator $s_\alpha$ corresponds to the reflection

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2\frac{(v|\alpha)}{(\alpha|\alpha)}\alpha$$

at a hyperplane perpendicular to the root vector $\alpha$.

The action of the Coxeter group is to permute these root vectors.
Form an algebra using the **Geometric Product**

\[ ab \equiv a \cdot b + a \wedge b \]

for two vectors

Extend via linearity and associativity to higher grade elements (multivectors)

For an \( n \)-dimensional space generated by \( n \) orthogonal unit vectors \( e_i \) have \( 2^n \) elements

Then \[ e_i e_j = e_i \wedge e_j = -e_j e_i \] so anticommute (Grassmann variables, exterior algebra)

Unlike the *inner* and *outer* products separately, this product is invertible
Basics of Clifford Algebra II

- These are known to have matrix representations over the normed division algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ → Classification of Clifford algebras
- E.g. Pauli algebra in 3D (likewise for Dirac algebra in 4D) is

$$\begin{align*}
\{1\} & \quad \{e_1, e_2, e_3\} & \quad \{e_1 e_2, e_2 e_3, e_3 e_1\} & \quad \{I \equiv e_1 e_2 e_3\} \\
\text{1 scalar} & \quad \text{3 vectors} & \quad \text{3 bivectors} & \quad \text{1 trivector}
\end{align*}$$

- These have the well-known matrix representations in terms of $\sigma$- and $\gamma$-matrices
- Working with these is not necessarily the most insightful thing to do, so here stress approach to work directly with the algebra
Clifford algebra is very efficient at performing reflections

Consider reflecting the vector $a$ in a hypersurface with unit normal $n$:

$$ a' = a_\perp - a_\parallel = a - 2a_\parallel = a - 2(a \cdot n)n $$

c.f. fundamental Weyl reflection $s_i : v \rightarrow s_i(v) = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$

But in Clifford algebra have $n \cdot a = \frac{1}{2}(na + an)$ so reassembles into (note doubly covered by $n$ and $-n$) sandwiching

$$ a' = -nan $$

So both Coxeter and Clifford frameworks are ideally suited to describing reflections – combine the two

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Platonic solids generate their four-dimensional analogues
Reflections and Rotations

- Generate a rotation when compounding two reflections wrt $n$ then $m$ (Cartan-Dieudonné theorem):

$$a'' = mnam \equiv Ra\tilde{R}$$

where $R = mn$ is called a spinor and a tilde denotes reversal of the order of the constituent vectors ($R\tilde{R} = 1$)

- All multivectors transform covariantly e.g.

$$MN \rightarrow (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$$

so transform double-sidedly

- Spinors form a group, which gives a representation of the Spin group $Spin(n)$ – they transform single-sidedly (obvious it’s a double (universal) cover)
Artin’s Theorem and orthogonal transformations

- **Artin**: every isometry is at most $d$ reflections
- Since have a double cover of reflections ($n$ and $-n$) we have a double cover of $O(p,q)$: $\text{Pin}(p,q)$
  \[
  x' = \pm n_1 n_2 \ldots n_k x n_k \ldots n_2 n_1
  \]
- **Pinors** = products of vectors $n_1 n_2 \ldots n_k$ encode orthogonal transformations via ‘sandwiching’
- **Cartan-Dieudonné**: rotations are an **even** number of reflections: $\text{Spin}(p,q)$ doubly covers $SO(p,q)$
Introduction
Combining Coxeter and Clifford

Coxeter groups and root systems
Clifford algebras
‘Platonic’ Solids

3D Platonic Solids

- There are 5 Platonic solids
- Tetrahedron *(self-dual) \((A_3)\)*
- Dual pair octahedron and cube \((B_3)\)*
- Dual pair icosahedron and dodecahedron \((H_3)\)*
- Only the octahedron is a root system (actually for \((A_1^3)\))*
Clifford and Coxeter: Platonic Solids

<table>
<thead>
<tr>
<th>Platonic Solid</th>
<th>Group</th>
<th>root system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>$A_3$</td>
<td>Cuboctahedron</td>
</tr>
<tr>
<td></td>
<td>$A_1^3$</td>
<td>Octahedron</td>
</tr>
<tr>
<td>Octahedron</td>
<td>$B_3$</td>
<td>Cuboctahedron + Octahedron</td>
</tr>
<tr>
<td>Cube</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Icosahedron</td>
<td>$H_3$</td>
<td>Icosidodecahedron</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td></td>
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</tr>
</tbody>
</table>

- **Platonic Solids** have been known for millennia
- Described by **Coxeter** groups
In 4D, there are 6 analogues of the Platonic Solids:
- **5-cell** (self-dual) \((A_4)\)
- **24-cell** (self-dual) \((D_4)\) – a 24-cell and its dual together are the \(F_4\) root system
- Dual pair **16-cell** and **8-cell** \((B_4)\)
- Dual pair **600-cell** and **120-cell** \((H_4)\)

These are 4D analogues of the **Platonic Solids**: regular convex 4-polytopes.
4D ‘Platonic Solids’

- 24-cell, 16-cell and 600-cell are all root systems, as is the related $F_4$ root system.
- 8-cell and 120-cell are dual to a root system, so in 4D out of 6 Platonic Solids only the 5-cell (corresponding to $A_n$ family) is not related to a root system!
- The 4D Platonic solids are not normally thought to be related to the 3D ones except for the boundary cells.
- They have very unusual automorphism groups.
- Some partial case-by-case algebraic results in terms of quaternions – here we show a uniform construction offering geometric understanding.
Mysterious Symmetries of 4D Polytopes

Spinorial symmetries

| rank 4   | $|\Phi|$ | Symmetry       |
|----------|---------|----------------|
| $D_4$ 24-cell | 24      | $2 \cdot 24^2 = 576$ |
| $F_4$ lattice | 48      | $48^2 = 2304$   |
| $H_4$ 600-cell | 120     | $120^2 = 14400$ |
| $A_1^4$ 16-cell | 8       | $3! \cdot 8^2 = 384$ |
| $A_2 \oplus A_2$ prism | 12      | $12^2 = 144$   |
| $H_2 \oplus H_2$ prism | 20      | $20^2 = 400$   |
| $I_2(n) \oplus I_2(n)$ | $2n$   | $(2n)^2$       |

Similar for Grand Antiprism ($H_4$ without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$).
Platonic Solids have been known for millennia; described by Coxeter groups.

Concatenating reflections gives Clifford spinors (binary polyhedral groups).

These induce 4D root systems

\[
\psi = a_0 + a_1 e_i \Rightarrow \psi \bar{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2
\]

4D analogues of the Platonic Solids and give rise to 4D Coxeter groups.
1 Introduction
- Coxeter groups and root systems
- Clifford algebras
- ‘Platonic’ Solids

2 Combining Coxeter and Clifford
- The Induction Theorem – from 3D to 4D
- Automorphism Groups
- Trinities and McKay correspondence
Theorem: 3D spinor groups give root systems.

Proof: 1. $R$ and $-R$ are in a spinor group by construction, 2. closure under reflections is guaranteed by the closure property of the spinor group

Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)

Counterexample: not every rank-4 root system is induced in this way
Induction Theorem – automorphism

- So induced 4D polytopes are actually root systems.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group.
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group.
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.
Recap: Clifford algebra and reflections & rotations

- Clifford algebra is very efficient at performing reflections via sandwiching
  \[ a' = -nan \]

- Generate a rotation when compounding two reflections wrt \( n \) then \( m \) (Cartan-Dieudonné theorem):
  \[ a'' = mnanm = Ra\tilde{R} \]

  where \( R = mn \) is called a spinor and a tilde denotes reversal of the order of the constituent vectors (\( R\tilde{R} = 1 \))
Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the Platonic Solids:
- The 6/12/18/30 reflections in $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$ generate 8/24/48/120 spinors.
- E.g. $\pm e_1, \pm e_2, \pm e_3$ give the 8 spinors $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The discrete spinor group is isomorphic to the quaternion group $Q$ / binary tetrahedral group $2T$ / binary octahedral group $2O$ / binary icosahedral group $2I$.

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Spinors and Polytopes

- The space of Cl(3)-spinors and quaternions have a 4D Euclidean signature: \( \psi = a_0 + a_i \epsilon_i \Rightarrow \psi \tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \)
- Can reinterpret spinors in \( \mathbb{R}^3 \) as vectors in \( \mathbb{R}^4 \)
- Then the spinors constitute the vertices of the 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell
- These are 4D analogues of the Platonic Solids: regular convex 4-polytopes
Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of $A_1 \times A_1 \times A_1 \times A_1$, $D_4$, $F_4$ and $H_4$

- Exceptional phenomena: $D_4$ (triality, important in string theory), $F_4$ (largest lattice symmetry in 4D), $H_4$ (largest non-crystallographic symmetry)

- Exceptional $D_4$ and $F_4$ arise from series $A_3$ and $B_3$

- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems
The spinors generated from the reflections contained in the respective rank-3 Coxeter group via the geometric product are realisations of the binary polyhedral groups $Q$, $2T$, $2O$ and $2I$, which were known to generate (mostly exceptional) rank-4 groups, but not known why, and why the ‘mysterious symmetries’.

<table>
<thead>
<tr>
<th>rank-3 group</th>
<th>diagram</th>
<th>binary</th>
<th>rank-4 group</th>
<th>diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 \times A_1 \times A_1$</td>
<td>o o o</td>
<td>$Q$</td>
<td>$A_1 \times A_1 \times A_1 \times A_1$</td>
<td>o o o o</td>
</tr>
<tr>
<td>$A_3$</td>
<td></td>
<td>$2T$</td>
<td>$D_4$</td>
<td></td>
</tr>
<tr>
<td>$B_3$</td>
<td>4</td>
<td>$2O$</td>
<td>$F_4$</td>
<td>4</td>
</tr>
<tr>
<td>$H_3$</td>
<td>5</td>
<td>$2I$</td>
<td>$H_4$</td>
<td>5</td>
</tr>
</tbody>
</table>
Only remaining case is what happens for $A_1 \oplus I_2(n)$ - this gives a doubling $I_2(n) \oplus I_2(n)$.

<table>
<thead>
<tr>
<th>rank 3</th>
<th>rank 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_3$</td>
<td>$D_4$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$F_4$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$H_4$</td>
</tr>
<tr>
<td>$A_1^3$</td>
<td>$A_1^4$</td>
</tr>
<tr>
<td>$A_1 \oplus A_2$</td>
<td>$A_2 \oplus A_2$</td>
</tr>
<tr>
<td>$A_1 \oplus H_2$</td>
<td>$H_2 \oplus H_2$</td>
</tr>
<tr>
<td>$A_1 \oplus I_2(n)$</td>
<td>$I_2(n) \oplus I_2(n)$</td>
</tr>
</tbody>
</table>
Automorphism Groups

- So induced 4D polytopes are actually root systems via the binary polyhedral groups.
- Clear why the number of roots $|\Phi|$ is equal to $|G|$, the order of the spinor group.
- Spinor group is trivially closed under conjugation, left and right multiplication. Results in non-trivial symmetries when viewed as a polytope/root system.
- Now explains symmetry of the polytopes/root system and thus the order of the rank-4 Coxeter group
- Theorem: The automorphism group of the induced root system contains two factors of the respective spinor group acting from the left and the right.
Spinorial Symmetries of 4D Polytopes

Spinorial symmetries

| rank 3 | $|\Phi|$ | $|W|$ | rank 4 | $|\Phi|$ | Symmetry |
|--------|--------|--------|--------|--------|----------|
| $A_3$  | 12     | 24     | $D_4$  | 24     | $2 \cdot 24^2 = 576$ |
| $B_3$  | 18     | 48     | $F_4$  | 48     | $48^2 = 2304$   |
| $H_3$  | 30     | 120    | $H_4$  | 120    | $120^2 = 14400$ |
| $A_1^3$| 6      | 8      | $A_1^4$| 8      | $3! \cdot 8^2 = 384$|
| $A_1 \oplus A_2$ | 8 | 12 | $A_2 \oplus A_2$ prism | 12 | $12^2 = 144$ |
| $A_1 \oplus H_2$ | 12 | 20 | $H_2 \oplus H_2$ prism | 20 | $20^2 = 400$ |
| $A_1 \oplus I_2(n)$ | $n + 2$ | $2n$ | $I_2(n) \oplus I_2(n)$ | $2n$ | $(2n)^2$ |

Similar for Grand Antiprism ($H_4$ without $H_2 \oplus H_2$) and Snub 24-cell ($2I$ without $2T$). Additional factors in the automorphism group come from 3D Dynkin diagram symmetries!
Some non-Platonic examples of spinorial symmetries

- **Grand Antiprism**: the 100 vertices achieved by subtracting 20 vertices of $H_2 \oplus H_2$ from the 120 vertices of the $H_4$ root system 600-cell – two separate orbits of $H_2 \oplus H_2$

- This is a semi-regular polytope with automorphism symmetry $\text{Aut}(H_2 \oplus H_2)$ of order $400 = 20^2$

- Think of the $H_2 \oplus H_2$ as coming from the doubling procedure? (Likewise for $\text{Aut}(A_2 \oplus A_2)$ subgroup)

- **Snub 24-cell**: $2T$ is a subgroup of $2I$ so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group $2T \times 2T$ of order $576 = 24^2$. 
Sub root systems

- The above spinor groups had spinor multiplication as the group operation.
- But also closed under \textit{twisted conjugation} – corresponds to closure under reflections (root system property).
- If we take \textit{twisted conjugation} as the group operation instead, we can have various subgroups.
- These are the remaining \textit{4D root systems} e.g. $A_4$ or $B_4$. 
Arnold’s Trinities

Arnold’s observation that many areas of real mathematics can be complexified and quaternionified resulting in theories with a similar structure.

- The fundamental trinity is thus \((\mathbb{R}, \mathbb{C}, \mathbb{H})\)
- The projective spaces \((\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)\)
- The spheres \((\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)\)
- The Möbius/Hopf bundles \((S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)\)
- The Lie Algebras \((E_6, E_7, E_8)\)
- The symmetries of the Platonic Solids \((A_3, B_3, H_3)\)
- The 4D groups \((D_4, F_4, H_4)\)
- New connections via my Clifford spinor construction (see McKay correspondence)
Platonic Trinities

- Arnold’s connection between \((A_3, B_3, H_3)\) and \((D_4, F_4, H_4)\) is very convoluted and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The number of Weyl chambers in each segment is:
  - \(24 = 2(1 + 3 + 3 + 5)\)
  - \(48 = 2(1 + 5 + 7 + 11)\)
  - \(120 = 2(1 + 11 + 19 + 29)\)
- Notice this miraculously matches the quasihomogeneous weights \(((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))\) of the Coxeter groups \((D_4, F_4, H_4)\)
- Believe the Clifford connection is more direct
A unified framework for polyhedral groups

<table>
<thead>
<tr>
<th>Group</th>
<th>Discrete subgroup</th>
<th>Action Mechanism</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO(3)$</td>
<td>rotational (chiral)</td>
<td>$x \rightarrow \tilde{R}xR$</td>
</tr>
<tr>
<td>$O(3)$</td>
<td>reflection (full/Coxeter)</td>
<td>$x \rightarrow \pm\tilde{A}xA$</td>
</tr>
<tr>
<td>$\text{Spin}(3)$</td>
<td>binary</td>
<td>$(R_1, R_2) \rightarrow R_1 R_2$</td>
</tr>
<tr>
<td>$\text{Pin}(3)$</td>
<td>pinor</td>
<td>$(A_1, A_2) \rightarrow A_1 A_2$</td>
</tr>
</tbody>
</table>

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar I** this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group $H_3$ in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building
Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate conjugacy classes etc of pinors in Clifford algebra
- Chiral (binary) polyhedral groups have irreps
  - tetrahedral (12/24): 1, 1', 1'', 2_s, 2', 2'', 3
  - octahedral (24/48): 1, 1', 2, 2_s, 2', 3, 3', 4_s
  - icosahedral (60/120): 1, 2_s, 2', 3, 3, 4, 4_s, 5, 6_s
- Binary groups are discrete subgroups of $SU(2)$ and all thus have a $2_s$ spinor irrep
- Connection with the McKay correspondence!
Affine extensions – $E_8^\infty$

AKA $E_8^+$ and along with $E_8^{++}$ and $E_8^{+++}$ thought to be the underlying symmetry of String and M-theory

Also interesting from a pure mathematics point of view: $E_8$ lattice, McKay correspondence and Monstrous Moonshine.
The McKay Correspondence

Exceptional Lie Groups
- $E_6$, 12
- $E_7$, 18
- $E_8$, 30
(Coxeter numbers)

Binary polyhedral groups
- $2T$, $2O$, $2I$
- $\sum d_i = 12, 18, 30$
- $\sum d_i^2 = 24, 48, 120$

Platonic solids generate their four-dimensional analogues
The McKay Correspondence

- Platonic Solids
  - 3D Coxeter groups
    - $A_3, B_3, H_3$
    - 12, 18, 30
  - PPD Clifford spinors

- Affine extensions, lattices

- Binary polyhedral groups
  - $2T, 2O, 2I$
  - $\sum d_i = 12, 18, 30$
  - $\sum d_i^2 = 24, 48, 120$
  - McKay correspondence

- Exceptional Lie Groups
  - $E_6, 12$
  - $E_7, 18$
  - $E_8, 30$
  - (Coxeter numbers)

- Monster, Baby Monster, Leech?

- PPD Clifford spinors

- 4D root systems
  - $D_4, 24^2$
  - $F_4, 48^2$
  - $H_4, 120^2, 30$

Platonic solids generate their four-dimensional analogues
More than E-type groups: the infinite family of 2D groups, the cyclic and dicyclic groups are in correspondence with $A_n$ and $D_n$, e.g. the quaternion group $Q$ and $D_4^+$. So McKay correspondence not just a trinity but ADE-classification. We also have $I_2(n)$ on top of the trinity ($A_3, B_3, H_3$)

<table>
<thead>
<tr>
<th>rank-3 group</th>
<th>diagram</th>
<th>binary</th>
<th>rank-4 group</th>
<th>diagram</th>
<th>Lie algebra</th>
<th>diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1 \times A_1 \times A_1$</td>
<td>$\circ \circ \circ$</td>
<td>$Q$</td>
<td>$A_1 \times A_1 \times A_1 \times A_1$</td>
<td>$\circ \circ \circ \circ$</td>
<td>$D_4^+$</td>
<td></td>
</tr>
<tr>
<td>$A_3$</td>
<td>$- - -$</td>
<td>$2T$</td>
<td>$D_4$</td>
<td>$- - -$</td>
<td>$E_6^+$</td>
<td></td>
</tr>
<tr>
<td>$B_3$</td>
<td>$- - -$</td>
<td>$4$</td>
<td>$F_4$</td>
<td>$- - -$</td>
<td>$E_7^+$</td>
<td></td>
</tr>
<tr>
<td>$H_3$</td>
<td>$- - -$</td>
<td>$5$</td>
<td>$H_4$</td>
<td>$- - -$</td>
<td>$E_8^+$</td>
<td></td>
</tr>
</tbody>
</table>
4D geometry is surprisingly important for HEP

- 4D root systems are surprisingly relevant to HEP
- $A_4$ is $SU(5)$ and comes up in Grand Unification
- $D_4$ is $SO(8)$ and is the little group of String theory
- In particular, its triality symmetry is crucial for showing the equivalence of RNS and GS strings
- $B_4$ is $SO(9)$ and is the little group of M-Theory
- $F_4$ is the largest crystallographic symmetry in 4D and $H_4$ is the largest non-crystallographic group
- The above are subgroups of the latter two
- Spinorial nature of the root systems could have surprising consequences for HEP
Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups
Advances in Applied Clifford Algebras, June 2013, Volume 23, Issue 2, pp 301-321

A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)


Platonic Solids generate their 4-dimensional analogues
Conclusions

- Novel *connection* between geometry of 3D and 4D
- In fact, 3D seems more *fundamental* – contrary to the *usual perspective* of 3D subgroups of 4D groups
- *Spinorial symmetries*
- Clear why *spinor group* gives a root system and why *two factors* of the same group reappear in the *automorphism group*
- Novel *spinorial perspective* on 4D geometry
- *Accidentalness* of the spinor construction and *exceptional* 4D phenomena
- Connection with Arnold’s *trinities*, the *McKay correspondence* and *Monstrous Moonshine*
Thank you!
Motivation: Viruses

- Geometry of polyhedra described by Coxeter groups
- Viruses have to be ‘economical’ with their genes
- Encode structure modulo symmetry
- Largest discrete symmetry of space is the icosahedral group
- Many other ‘maximally symmetric’ objects in nature are also icosahedral: Fullerenes & Quasicrystals
- But: viruses are not just polyhedral – they have radial structure. Affine extensions give translations
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon
Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon

Platonic solids generate their four-dimensional analogues
Unit translation along a vertex of a unit pentagon

A random translation would give 5 secondary pentagons, i.e. 25 points. Here we have degeneracies due to ‘coinciding points’.
Affine extensions of non-crystallographic root systems

Translation of length $\tau = \frac{1}{2} (1 + \sqrt{5}) \approx 1.618$ (golden ratio)

Looks like a virus or carbon onion
Extend icosahedral group with distinguished translations

- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the outer shape (capsid)
- Can predict inner structure (nucleic acid distribution) of the virus from the point array

Affine extensions of the icosahedral group (giving translations) and their classification.
Use in Mathematical Virology

- Suffice to say point arrays work very exceedingly well in practice. Two papers on the mathematical (Coxeter) aspects.
- Implemented computational problem in Clifford – some very interesting mathematics comes out as well (see later).
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions \((C_{60} - C_{240} - C_{540})\)
Extension to fullerenes: carbon onions

- Extend idea of affine symmetry to other icosahedral objects in nature: football-shaped fullerenes
- Recover different shells with icosahedral symmetry from affine approach: carbon onions \((C_{80} - C_{180} - C_{320})\)
References

- Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups with Twarock/Bœhm

- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bœhm
  Journal of Mathematical Physics 54 093508 (2013), Cover article September

There are interesting applications to quasicrystals, viruses or carbon onions, but here concentrate on the mathematical aspects.
Quaternions and Clifford Algebra

- The unit spinors \( \{1; le_1; le_2; le_3\} \) of \( \text{Cl}(3) \) are isomorphic to the quaternion algebra \( \mathbb{H} \) (up to sign)
- The 3D Hodge dual of a vector is a pure bivector which corresponds to a pure quaternion, and their products are identical (up to sign)
Discrete Quaternion groups

- The 8 quaternions of the form \((\pm 1, 0, 0, 0)\) and permutations are called the Lipschitz units, and form a realisation of the quaternion group in 8 elements.

- The 8 Lipschitz units together with \(\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)\) are called the Hurwitz units, and realise the binary tetrahedral group of order 24. Together with the 24 ‘dual’ quaternions of the form \(\frac{1}{\sqrt{2}}(\pm 1, \pm 1, 0, 0)\), they form a group isomorphic to the binary octahedral group of order 48.

- The 24 Hurwitz units together with the 96 unit quaternions of the form \((0, \pm \tau, \pm 1, \pm \sigma)\) and even permutations, are called the Icosians. The icosian group is isomorphic to the binary icosahedral group with 120 elements.
Groups $E_8$, $D_4$, $F_4$ and $H_4$ have representations in terms of quaternions

Extensively used in the high energy physics/quasicrystal/Coxeter/polytope literature and thought of as deeply significant, though not really clear why

e.g. $H_4$ consists of 120 elements of the form $(\pm 1, 0, 0, 0)$, $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$ and $(0, \pm \tau, \pm 1, \pm \sigma)$

Seen as remarkable that the subset of the 30 pure quaternions is a realisation of $H_3$ (a sub-root system)

Similarly, $A_3$, $B_3$, $A_1 \times A_1 \times A_1$ have representations in terms of pure quaternions

Will see there is a much simpler geometric explanation
Quaternionic representations used in the literature

\[ A_1 \times A_1 \times A_1 \]
\[ A_3 = D_3 \]
\[ B_3 \]
\[ H_3 \]
\[ F_4 \]
\[ D_4 \]
Demystifying Quaternionic Representations

- **3D**: Pure quaternions = Hodge dualised (pseudoscalar) root vectors
- **In fact, they are the simple roots of the Coxeter groups**
- **4D**: Quaternions = disguised spinors – but those of the 3D Coxeter group i.e. the binary polyhedral groups!
- **This relation between 3D and 4D via the geometric product does not seem to be known**
- Quaternion multiplication = ordinary Clifford reflections and rotations
Demystifying Quaternionic Representations

- Pure quaternion subset of 4D groups only gives 3D group if the 3D group contains the inversion/pseudoscalar $i$
- e.g. does not work for the tetrahedral group $A_3$, but $A_3 \rightarrow D_4$ induction still works, with the central node essentially ‘spinorial’
- In fact, it goes the other way around: the 3D groups induce the 4D groups via spinors
- The rank-4 groups are also generated (under quaternion multiplication) by two quaternions we can identify as $R_1 = \alpha_1 \alpha_2$ and $R_2 = \alpha_2 \alpha_3$
- Can see these are ‘spinor generators’ and how they don’t really contain any more information/roots than the rank-3 groups alone
Sandwiching is often seen as particularly nice feature of the quaternions giving rotations

This is actually a general feature of Clifford algebras/versors in any dimension; the isomorphism to the quaternions is accidental to 3D

However, the root system construction does not necessarily generalise

2D generalisation merely gives that $l_2(n)$ is self-dual

Octonionic generalisation just induces two copies of the above 4D root systems, e.g. $A_3 \rightarrow D_4 \oplus D_4$