Quadrangle groups inclusions

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Motivation

Triangle groups: $\Delta(k,m,n) = \langle x,y,z \mid x^k, y^m, z^n, xyz \rangle$

Finite index inclusions (D. Singerman 1972):

Families:

$\Delta(k,n,n) \triangleleft_2 \Delta(2,2k,n)$
$\Delta(n,n,n) \triangleleft_3 \Delta(3,3,n)$
$\Delta(2,n,2n) <_3 \Delta(2,3,2n)$
$\Delta(3,n,3n) <_4 \Delta(2,3,3n)$
$\Delta(n,2n,2n) <_4 \Delta(2,4,2n)$
$\Delta(n,4n,4n) <_6 \Delta(2,3,4n)$
$\Delta(n,n,n) \triangleleft_6 \Delta(2,3,2n)$

Sporadic:

$\Delta(4,4,5) <_6 \Delta(2,4,5)$
$\Delta(3,3,7) <_8 \Delta(2,3,7)$
$\Delta(2,7,7) <_9 \Delta(2,3,7)$
$\Delta(3,8,8) <_{10} \Delta(2,3,8)$
$\Delta(4,8,8) <_{12} \Delta(2,3,8)$
$\Delta(9,9,9) <_{12} \Delta(2,3,9)$
$\Delta(7,7,7) <_{24} \Delta(2,3,7)$

Quadrangle groups:

$Q(k,\ell,m,n) = \langle x,y,z,w \mid x^k, y^\ell, z^m, w^n, xyzw \rangle$

Finite index inclusions?

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$$\Delta(3, n, 3n) <_4 \Delta(2, 3, 3n)$$
$$\Delta(n, 2n, 2n) <_4 \Delta(2, 4, 2n)$$
$$\Delta(n, 4n, 4n) <_6 \Delta(2, 3, 4n)$$
$$\Delta(n, n, n) \triangleleft_6 \Delta(2, 3, 2n)$$

Sporadic:

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Quadrangle groups:

$$Q(k, \ell, m, n) = \langle x, y, z, w \mid x^k, y^\ell, z^m, w^n, xyzw \rangle$$

Finite index inclusions?

Some known facts

- $Q := Q(k, \ell, m, n) < PSL(2, \mathbb{R})$ is a Fuchsian group.
- $k = ord(x)$, $\ell = ord(y)$, $m = ord(z)$ and $n = ord(w)$ are called the periods of Q.
- $$Q(k, \ell, m, n) \cong Q(\sigma(k), \sigma(\ell), \sigma(m), \sigma(n))$$

  for any permutation σ of $\{k, \ell, m, n\}$.

Remarks

- We can assume $k \leqslant \ell \leqslant m \leqslant n$, up to isomorphism.
- $Q(1, \ell, m, n) = \Delta(\ell, m, n)$.

Riemann-Hurwitz (RH)

$\boxed{Q' := \langle a, b, c, d \mid a^p, b^q, c^r, d^s, abcd \rangle}$ subgroup of

$\boxed{Q := \langle x, y, z, w \mid x^k, y^\ell, z^m, w^n, xyzw \rangle}$ of index N.

We write $Q' <_N Q$ or $Q(p, q, r, s) <_N Q(k, \ell, m, n)$. Then

$$\boxed{N(2 - \Sigma) = 2 - \Sigma'} \qquad \text{(RH)}$$

$$\Sigma := \frac{1}{k} + \frac{1}{\ell} + \frac{1}{m} + \frac{1}{n}, \qquad \Sigma' := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}.$$

Remarks

1) $\Sigma > 2 \Leftrightarrow \Sigma' > 2$, $\Sigma = 2 \Leftrightarrow \Sigma' = 2$, $\Sigma < 2 \Leftrightarrow \Sigma' < 2$

2) If $\Sigma < 2$, then $N = \frac{2-\Sigma'}{2-\Sigma} < \frac{2}{2-\max\Sigma} = 84$ (RH-bound).
$\max\Sigma = \frac{83}{42}$ reached by $(k, \ell, m, n) = (1, 2, 3, 7)$. Moreover, (RH) $\Rightarrow k < 4$ (later $k \neq 3$ and hence $k = 1$ or $k = 2$).

- If $\Sigma > 2$ then Q is cyclic, dihedral, $A_4 = \Delta(2,3,3)$, $S_4 = \Delta(2,3,4)$ or $A_5 = \Delta(2,3,5)$ and so do Q'.
- If $\Sigma = 2$, then $(k, \ell, m, n) \in \{(1,2,3,6), (1,2,4,4), (1,3,3,3), (2,2,2,2)\}$ and Q contains copies of itself as subgroups of finite index ($Q' = Q$).

From now on $\boxed{\Sigma < 2}$. Then Q is not finite and any element of finite order in Q is conjugate to a power of one and only one of the generators x, y, z, w. We write $g \sim x$ if g is a conjugate of a power of x.

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- If $\Sigma = 2$, then $(k,\ell,m,n) \in \{(1,2,3,6),(1,2,4,4),(1,3,3,3),(2,2,2,2)\}$ and Q contains copies of itself as subgroups of finite index ($Q' = Q$).

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Case subdivision

In particular, any of the generators a, b, c, d of Q' is a conjugate of a power of x, y, z or w.

- **Case 1:** $\boxed{a, b, c, d \sim w}$
- **Case 2:** $a, b, c, d \sim z, w$. Split in
  - **Case 2i:** $\boxed{a \sim z \quad \text{and} \quad b, c, d \sim w}$
  - **Case 2ii:** $\boxed{a, b \sim z \quad \text{and} \quad c, d \sim w}$
- **Case 3:** $\boxed{a \sim y, \quad b \sim z \quad \text{and} \quad c, d \sim w}$

There is no **Case 4**: $N = \frac{2-\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{1}{s}\right)}{2-\left(\frac{1}{k}+\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}\right)} > 1$.

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Case 1

Case 1: $a, b, c, d \sim w$.

From Singerman (1970):

There is an epimorphism θ from Q to a finite permutation group G transitive on N points (cosets of Q' in Q) such that the constellation $\mathcal{C} = (\theta(x), \theta(y), \theta(z), \theta(w))$ has passport

$$\left(k^{\frac{N}{k}}, \ell^{\frac{N}{\ell}}, m^{\frac{N}{m}}, \frac{n}{p} \cdot \frac{n}{q} \cdot \frac{n}{r} \cdot \frac{n}{s} \cdot n^{\frac{N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right)}{n}}\right) \qquad \text{(P1)}$$

Remarks

- G is the cartographic group (monodromy group) of $\mathcal{C}$, which has genus 0.
- If $k = 1$ then $Q = \Delta(\ell, m, n)$ and $(\theta(y), \theta(z))$ is a dessin d'enfant (hypermap) $\longrightarrow$ picture.
- If G has size N ($\mathcal{C}$ is regular), then $Q' \lhd_N Q$.

Hence

- k, ℓ, m divide N,
- p, q, r, s divide n,
- n divide $N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right)$,

and (RH) is equivalent to

$$\boxed{(2N-2) - \frac{N}{k} - \frac{N}{\ell} - \frac{N}{m} = \frac{N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right)}{n}} \qquad \text{(RH1)}$$

Remark

Solutions of (RH1) with $N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right) = 0$ give rise to **families of inclusions.**
Solutions of (RH1) with $N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right) \neq 0$ give rise to **sporadic inclusions.**

Examples

Families: $N = \frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s} \Leftrightarrow 2N - 2 = \frac{N}{k} + \frac{N}{\ell} + \frac{N}{m} \Rightarrow$ $(k, \ell, m; N)$ is one of the following

$(2,2,2;4), (1,4,4;4), (1,3,3;6), (1,2,6;6), (1,2,4;8), (1,2,3;12)$

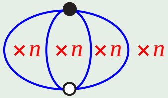
Example $(k, \ell, m; N) = (2,2,2;4)$

Then $4 = N = \frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s} \Rightarrow p = q = r = s = n$, giving the passport $(2^2, 2^2, 2^2, 1^4)$ of the constellation $(x, y, z, w) = ((12)(34), (13)(24), (14)(23), 1)$ with cartographic group $C_2 \times C_2$, proving the normal inclusion

$$\boxed{Q(n,n,n,n) \lhd_4 Q(2,2,2,n)}$$

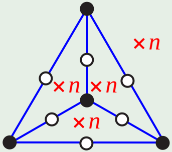
Example $(k, \ell, m; N) = (1, 4, 4; 4)$

Again $p = q = r = s = n$ giving the passport $(1^4, 4, 4, 1^4)$ of the dessin d'enfant:



proving the inclusion $\boxed{Q(n, n, n, n) \lhd_4 \Delta(4, 4, n)}$

Example



$\boxed{Q(n, n, n, n) \lhd_{12} \Delta(2, 3, 3n)}$

Remark: Here $(k, \ell, m; N) = (1, 2, 3; 12)$ and $12 = N = 3 + 3 + 3 + 3$ (we replace n by $3n$).

Example $(k, \ell, m; N) = (1, 4, 4; 4)$

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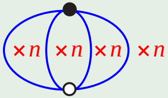

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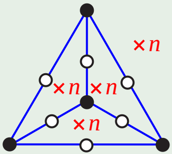

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Remark: Here $(k, \ell, m; N) = (1, 2, 3; 12)$ and $12 = N = 3 + 3 + 3 + 3$ (we replace n by $3n$).

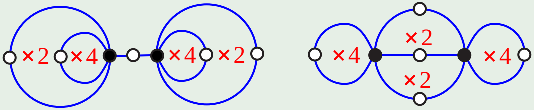
Example

For $(k, \ell, m; N) = (1, 2, 3; 12)$ with $12 = N = 7 + 3 + 1 + 1$ we get the passport $(1^{12}, 2^6, 3^4, 1^2 \cdot 3 \cdot 7)$ belonging to no constellation. $\Rightarrow Q(3n, 7n, 21n, 21n) \not<_{12} \Delta(2, 3, 21n)$.

Sporadic: See which positive values v the left hand side of (RH1) takes for different N, k, ℓ and m and solve

$$v = \frac{N - \left(\frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s}\right)}{n} \Leftrightarrow N = \frac{n}{p} + \frac{n}{q} + \frac{n}{r} + \frac{n}{s} + vn\,.$$

Example ($N = 10$, $k = 1$, $\ell = 2$ and $m = 5 \Rightarrow v = 1$)



$$\boxed{Q(2, 2, 4, 4) <_{10} \Delta(2, 5, 4)}$$

Remark: Same monodromy group.

Case 2i: $a \sim z$ and $b, c, d \sim w$.

From Singerman (1970):

There is a constellation with passport

$$\left(k^{\frac{N}{k}}, \ell^{\frac{N}{\ell}}, \frac{m}{p} \cdot m^{\frac{N-\frac{m}{p}}{m}}, \frac{n}{q} \cdot \frac{n}{r} \cdot \frac{n}{s} \cdot n^{\frac{N-\left(\frac{n}{q}+\frac{n}{r}+\frac{n}{s}\right)}{n}} \right) \qquad \text{(P2i)}$$

and (RH) is equivalent to

$$\boxed{(2N-2) - \frac{N}{k} - \frac{N}{\ell} = \frac{N-\frac{m}{p}}{m} + \frac{N-\left(\frac{n}{q}+\frac{n}{r}+\frac{n}{s}\right)}{n}} \qquad \text{(RH2i)}$$

Case 2ii

Case 2ii: $a, b \sim z$ and $c, d \sim w$.

From Singerman (1970):

There is a constellation with passport

$$\left(k^{\frac{N}{k}}, \ell^{\frac{N}{\ell}}, \frac{m}{p} \cdot \frac{m}{q} \cdot m^{\frac{N-(\frac{m}{p}+\frac{m}{q})}{m}}, \frac{n}{r} \cdot \frac{n}{s} \cdot n^{\frac{N-(\frac{n}{r}+\frac{n}{s})}{n}}\right) \qquad \text{(P2ii)}$$

and (RH) is equivalent to

$$\boxed{(2N-2) - \frac{N}{k} - \frac{N}{\ell} = \frac{N-(\frac{m}{p}+\frac{m}{q})}{m} + \frac{N-\left(\frac{n}{r}+\frac{n}{s}\right)}{n}} \qquad \text{(RH2ii)}$$

Case 3: $a \sim y, \quad b \sim z \quad$ and $\quad c, d \sim w$.

From Singerman (1970):

There is a constellation with passport

$$\left(k^{\frac{N}{k}}, \frac{\ell}{p} \cdot \ell^{\frac{N-\frac{\ell}{p}}{\ell}}, \frac{m}{q} \cdot m^{\frac{N-\frac{m}{q}}{m}}, \frac{n}{r} \cdot \frac{n}{s} \cdot n^{\frac{N-\left(\frac{n}{r}+\frac{n}{s}\right)}{n}}\right) \tag{P3}$$

and (RH) is equivalent to

$$\boxed{(2N-2) - \frac{N}{k} = \frac{N-\frac{\ell}{p}}{\ell} + \frac{N-\frac{m}{q}}{m} + \frac{N-\left(\frac{n}{r}+\frac{n}{s}\right)}{n}} \tag{RH3}$$