Splittable and unsplittable graphs and configurations

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Cyclic Haar graphs

Definition (Cyclic Haar graph)

Let $S \subseteq \mathbb{Z}_k$. The graph with vertex set $\{u_i, v_i \mid i \in \mathbb{Z}_k\}$ and edge set $\{u_i v_{i+\ell} \mid i \in \mathbb{Z}_n, \ell \in S\}$, denoted $H(k, S)$, is called a cyclic Haar graph of $\mathbb{Z}_k$ with respect to symbol S.

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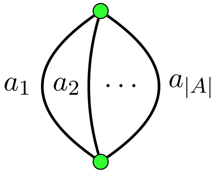
More general definition:

Definition (Haar graph)

Let Γ be an abelian group, $A \subseteq \Gamma$. A dipole with $|A|$ parallel arcs, labeled by elements of $A = \{a_1, a_2, \ldots\}$ is a voltage graph. Its regular covering graph, denoted $H(\Gamma, A)$, is called a Haar graph.



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(Let $n = \sum_i b_i 2^i$.) Define $H(n) = H(1 + \lfloor \log_2 n \rfloor, \{i \mid b_i = 1\})$.

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Note: There may exist $n_1 \neq n_2$ such that $H(n_1) \cong H(n_2)$. Smallest such number is called canonical number.

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Some more facts . . .

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Proposition

Cubic connected cyclic Haar graphs are hamiltonian.

(Alspach and Zhang proved that every cubic Cayley graph of a dihedral group is hamiltonian.)

Proposition

Let $H(n)$ be a connected cyclic Haar graph. Then one of the following is true:

1. $n = 1$ *and* $H(1) \cong K_2$ *has infinite girth;*
2. $n = 2^{k-1} + 1$ *and* $H(n) \cong C_{2k}$ *has girth* $2k$*;*
3. $H(n)$ *has valency greater than 2 and girth 4;*
4. $H(n)$ *has valency greater than 2 and girth 6.*

Configurations

Definition

A combinatorial (v_k) configuration is an incidence structure $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, $\mathcal{P} \cap \mathcal{B} = \emptyset$, where:

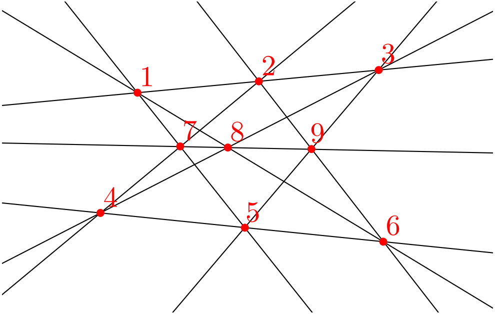
1. $|\mathcal{P}| = |\mathcal{B}| = v$,
2. $|\{b \mid (p, b) \in \mathcal{I}\}| = k$ for every $p \in \mathcal{P}$ (i.e. there are k lines through each point), and
3. $|\{p \mid (p, b) \in \mathcal{I}\}| = k$ for every $b \in B$ (i.e. there are k points on each line).

- The elements of $\mathcal{P}$ are called points.
- The elements of $\mathcal{B}$ are called lines (sometimes blocks).
- The relation $\mathcal{I}$ is called incidence.

Comment: There may be only one line going through two different points.

Configurations

An example



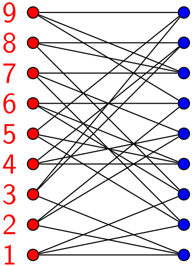
Configuration table:

1	4	7	1	1	2	2	3	3
2	5	8	5	6	4	6	4	5
3	6	9	7	8	7	9	8	9

Levi graph

Definition

The bipartite graph $L(\mathcal{C})$ on the vertex set $\mathcal{P} \cup \mathcal{B}$ with edges between $p \in \mathcal{P}$ and $b \in \mathcal{B}$ if the elements p and b are incident in $\mathcal{C}$, i.e. if $(p, b) \in \mathcal{I}$, is called the Levi graph of configuration $\mathcal{C}$.



Note: Any configuration is completely determined by a k-valent 2-colored graph of girth at least 6.

Isomorphism. Dual configuration

Definition

An isomorphism between $\mathcal{C}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\mathcal{C}_2 = (\mathcal{P}_2, \mathcal{B}_2, \mathcal{I}_2)$ is a bijective map $\alpha : \mathcal{P}_1 \cup \mathcal{B}_1 \to \mathcal{P}_2 \cup \mathcal{B}_2$, $\alpha(\mathcal{P}_1) \subseteq \mathcal{P}_2$, $\alpha(\mathcal{B}_1) \subseteq \mathcal{B}_2$, such that

$$(p, b) \in \mathcal{I}_1 \iff (\alpha(p), \alpha(b)) \in \mathcal{I}_2$$

for every $p \in \mathcal{P}_1$ and every $b \in \mathcal{B}_1$.

Definition

Configuration $\mathcal{C}^* = (\mathcal{B}, \mathcal{P}, \mathcal{I}^{-1})$ is called the dual of configuration $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$.

Reverse coloring of vertices of the Levi graph determines the dual configuration.

A configuration that is isomorphic to its dual is called self-dual.

Cyclic configurations

Definition

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Cyclic configurations and cyclic Haar graphs are closely related:

Corollary (Hladnik, Marušič, Pisanski)

1. *The cyclic Haar graphs of girth 6 are precisely Levi graphs of cyclic configurations.*
2. *Each cyclic configuration is self-dual, point-transitive, and line-transitive.*
3. *There are no triangle-free cyclic configurations.*

Square graph

Definition

The square of graph G, denoted G^2, is a graph with vertex set $V(G^2) = V(G)$ where two vertices are adjacent if and only if their distance in G is at most 2, i.e. $E(G^2) = \{uv \mid d_G(u, v) \leq 2\}$.

The square of a Levi graph $L(\mathcal{C})$ is called the Grünbaum graph of $\mathcal{C}$.

Unsplittable configurations

Formally introduced in the monograph *Configurations of Points and Lines* by Grünbaum. Later also used in *Configurations from a Graphical Viewpoint* by Servatius and Pisanski.

Definition

A configuration $\mathcal{C}$ is splittable if there exists an independent set of vertices S in the Grünbaum graph $L^2(\mathcal{C})$ such that the graph obtained by removing S from the Levi graph $L(\mathcal{C})$ is disconnected.

Set S is called a splitting set of elements. A configuration that is not splittable is called unsplittable.

An independent set in the Grünbaum graph is called independent set of elements of $\mathcal{C}$.

The Pappus configuration is unsplittable.

Unsplittable graphs

The notion was generalized to graphs by T. W. Tucker and Pisanski.

Definition

A graph G is splittable if there exists an independent set S in G^2 such that $X - S$ is disconnected.

Maximum number of independent elements

Grünbaum's conjecture disproved

Grünbaum conjectured and upper bound of $\lfloor v/r \rfloor + 1$ for the size of a maximal independent set of elements of $\mathcal{C}$.

Theorem (Tucker, Pisanski)

Let G be a r-regular graph on n vertices and let M be the size of a maximal independent set of G^2. Then

$$M \leq \lfloor n/(r+1) \rfloor.$$

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Theorem (Tucker, Pisanski)

Let M be the size of an independent set of elements of a (v_r) configuration. Then

$$M \leq \lfloor 2v/(r+1) \rfloor.$$

Moreover, for each integer $r \geq 3$, there exists an integer v, divisible by $r+1$, and a connected geometric (v_r) configuration with $M = 2v/(r+1)$.

Refinement of (un)splittability

Grünbaum also considered refinements of the notion of splittability:

Definition

Configuration $\mathcal{C}$ is point-splittable (line-splittable) if it is splittable and the splitting set of elements consists of points only (lines only).

Note: These refinements can be defined for any 2-colored graph.

For a configuration there are four possibilities – splitting types:
Any configuration may be:

- Type 1: point-splittable, line-splittable
- Type 2: point-splittable, line-unsplittable
- Type 3: point-unsplittable, line-splittable
- Type 4: point-unsplittable, line-unsplittable

Note: Configurations of splitting types 1, 2, and 3 are splittable.

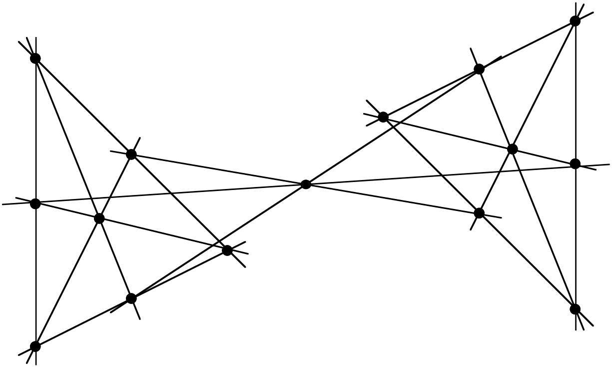

Figure: A point-splittable configuration of type 2. Its dual is of type 3.

Note: A configuration of splitting type 4 (point-unsplittable, line-unsplittable) may be splittable or unsplittable.

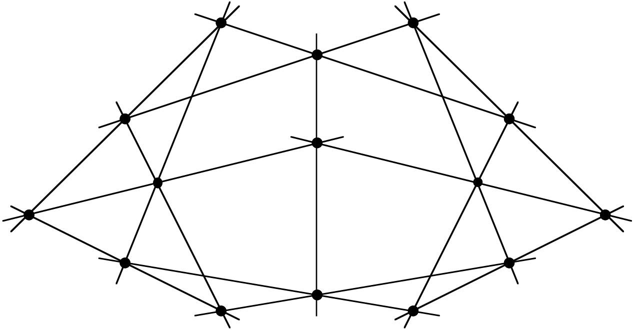

Figure: A splittable configuration of type 4.

Proposition

If $\mathcal{C}$ is of type 1 then its dual is of type 1. If it is of type 2 then its dual is of type 3. If it is of type 4 then its dual is of type 4.

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This has a straightforward consequence for cyclic configurations:

Corollary

Any self-dual configuration (in particular any cyclic configuration) is either of type 1 or 4.

- In 3-valent case combinatorial isomorphisms of cyclic configurations are well-understood.
- One would expect that large sparse graphs are splittable. In this sense the following result is not a surprise:

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There exist infinitely many cyclic (v_3) configurations that are splittable.

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Use cyclic Haar graphs $H(v, \{0, 1, 4\})$, where $v \geq 13$.

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(We also found other families of splittable cyclic Haar graphs with girth 6, e.g. $H(v, \{0, 1, 5\})$ and $H(v, \{0, 2, 5\})$ for $v \geq 16$.)

Splittable and unsplittable cyclic configurations

How about unsplittable configurations?

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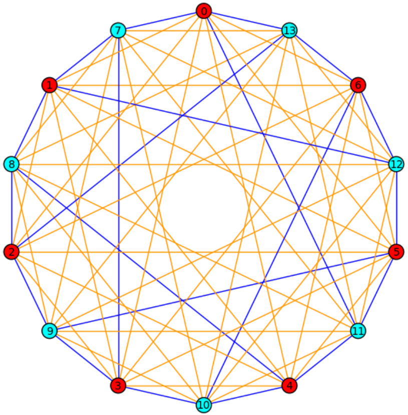
There exist infinitely many cyclic (v_3) configurations that are unsplittable.

Use the cyclic Haar graphs $H(v, \{0, 1, 3\}) = \mathrm{LCF}[5, -5]^n$, where $n \geq 7$.

Figure: $H(7, \{0, 1, 3\}) = \mathrm{CLF}[5, -5]^7$ alias the Heawood graph (blue edges) and its Grünbaum graph (blue and orange).



There is another infinite family:

Proposition

Cyclic configurations defined by $H(3n, \{0, 1, n\})$*, where* $n \geq 2$*, are unsplittable.*

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We're working on complete characterization of cyclic (v_3) configurations with respect to splittability. We believe there are just two more families apart from those mentioned above . . .

Complete list of cyclic Haar graphs (up to 30 vertices)

n	all	girth 6	split.	unsplit.	split., g. 6	unsplit., g. 6
3	1	0	0	1	0	0
4	1	0	0	1	0	0
5	1	0	0	1	0	0
6	2	0	0	2	0	0
7	2	1	0	2	0	1
8	3	1	1	2	0	1
9	2	1	0	2	0	1
10	3	1	1	2	0	1
11	2	1	0	2	0	1
12	5	3	1	4	0	3
13	3	2	1	2	1	1
14	4	2	2	2	1	1
15	5	4	1	4	1	3
16	5	3	3	2	2	1

List of cyclic Haar graphs (up to 30 vertices) cont'd

n	all	girth 6	split.	unsplit.	split., g. 6	unsplit., g. 6
17	3	2	1	2	1	1
18	6	4	3	3	2	2
19	4	3	2	2	2	1
20	7	5	5	2	4	1
21	7	6	3	4	3	3
22	6	4	4	2	3	1
23	4	3	2	2	2	1
24	11	9	7	4	6	3
25	5	4	3	2	3	1
26	7	5	5	2	4	1
27	6	5	3	3	3	2
28	9	7	7	2	6	1
29	5	4	3	2	3	1
30	13	11	9	4	8	3

To be continued . . .