MST209
Bridging Material

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Introduction and study guide

MST209 is designed to follow on from MS221. However, we realize that some degree profiles do not include MS221, So we have designed this Bridging Material to help bridge the gap between MST121 and MST209. Most of the topics are covered in MST121 with the exception of two — complex numbers and Taylor polynomials. In addition you will find that MST209 assumes that you can routinely differentiate and integrate more complicated functions than those in MST121.

As you are studying this Bridging Material, you will already know something about calculus. However MST209 assumes a rather deeper knowledge than that given by an introductory calculus course and also it requires you to have some facility with calculation and manipulation. The aim of this material is to give you the necessary knowledge and to teach the techniques you will need. Throughout the three sections there are numerous exercises and it is important that you work through these until you feel confident that you have mastered the particular topic. No one learns mathematics solely by reading — the secret is practice. Once you have worked through several examples you should find that the ideas begin to 'stick' and you can then move on to the next topic.

You may have studied MST121, which introduces most of the calculus required for MST209.

If you have recently completed an introductory calculus course you should find that you can study all this material comfortably in two to three weeks. You may find some of it easy — in which case do not work slavishly through every detail. However, if you find parts of it unfamiliar, we suggest that you attempt all the exercises.

For example, you may have just completed MST121.

When you receive the MST209 course materials you will find that a computer algebra package is included. This is used throughout the course. Much of the work of these three sections can be performed using this package and the more difficult examples might better be tackled this way. Nevertheless you still need to be able to do simple calculations 'by hand' and working through these three sections will help to give you the skills you will need.

The first section covers differentiation and some of its applications. We assume that you have some knowledge of differentiation and so the basics are described only briefly. We then go on to extend your knowledge of differentiation and to show you how to differentiate some of the more complicated functions that will crop up in MST209.

The second section follows the same path with integration, and introduces two important techniques.

Section 3 introduces you to complex numbers, including their algebraic and geometric representations.

1 Differentiation

This section builds on your previous experiences of differentiation and explores some ideas in greater depth. You should aim to increase the range of functions that you are able to differentiate by hand by working through the examples and exercises.

There are various applications of differentiation used in MST209 in developing and analysing mathematical models. For example, it is frequently necessary to find the maximum or minimum value of a function, or to approximate a complicated function by a simpler one. These ideas are also covered in this section.

We begin by stating the definition of the *derivative* of a function at a point.

Definition

If $f(x)$ is a smooth function, then the corresponding **derived function** or **derivative** $f'(x)$ is defined by

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right).$$

A *smooth* function is one whose graph is a smooth curve, with no jumps or sharp changes of direction.

Notice that $f'(x)$ is a *function* and as x varies it will probably take different values.

The independent variable does not have to be x. It may be denoted by any letter. When the independent variable is time, the letter t is usually used.

Another way of thinking about the derivative is to look at the process of finding the gradient of a curve $y = f(x)$. The gradient of a curve at a point is defined to be the gradient of a tangent to the curve at that point (see Figure 1.1).

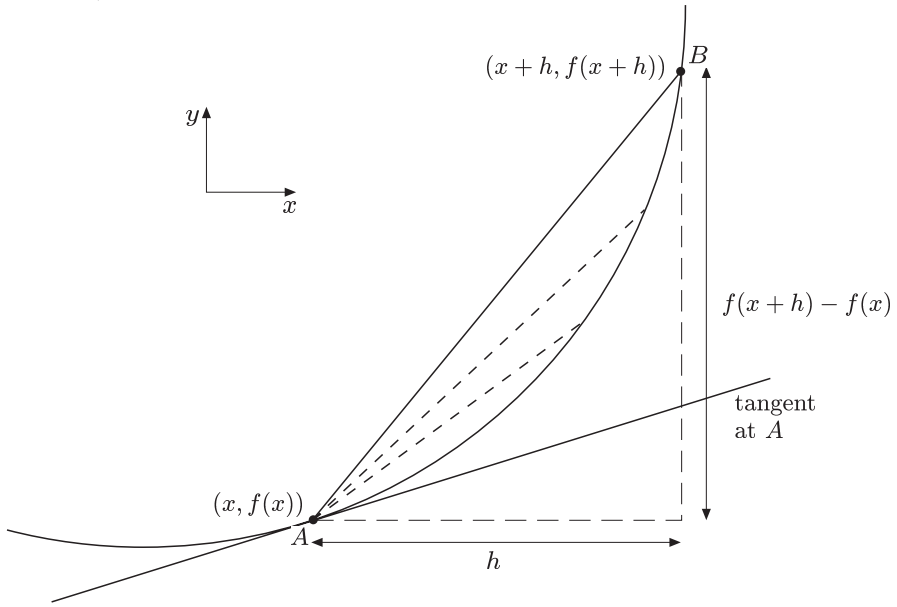


Figure 1.1

The gradient of a chord joining the points A and B in fIGURE 1.1 is

$$\frac{\text{increase in the value of } y}{\text{increase in the value of } x}.$$

If the increase in x from A to B is h, then the corresponding increase in y is $f(x+h) - f(x)$, and so the gradient of the chord is $(f(x+h) - f(x))/h$.

As B approaches A, the chord approaches the tangent at A, so

$$\begin{aligned}\text{gradient of curve at } A &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\\ &= \text{instantaneous rate of change of } f(x) \text{ at } A\\ &= f'(x),\end{aligned}$$

which is the derivative of $f(x)$ at A.

If the curve is a distance–time curve, then x will be replaced by t (representing time) and the gradient of the curve will give the velocity at that point.

There are three main notations commonly used for derivatives.

- The notation you have already seen here, which uses a prime (or dash): the derivative of $f(x)$ is $f'(x)$.
- Leibniz notation, which takes the form dy/dx, where y is the dependent variable and x is the independent variable. This is called the derivative of y with respect to x: if $y = f(x)$, then $dy/dx = f'(x)$.
- Newton's notation, which is used only when the independent variable is time, and consists of putting a dot over the dependent variable to indicate that it is the derivative with respect to time: $\dot{s}$ in place of ds/dt.

This notation is not used in this Bridging Material, but it will be used extensively in the MST209 course units.

Subsection 1.1 revises the techniques for differentiating simple functions by hand. In Subsection 1.2 we extend the rules for differentiation so that you can differentiate products, quotients and composites of these simple functions. You will then be able to differentiate by hand such expressions as $e^{3x}\cos(4x)$, $(x^3+1)/\sin(2x)$ and $\cos(x^3+1)$. Subsection 1.3 looks at differentiating inverse functions, notably the logarithm function. Finally, Subsection 1.4 introduces Taylor polynomials, which provide a useful way of approximating a function by a polynomial.

If you have just finished MST121 then you may be able to study Subsection 1.1 very quickly.

1.1 Revising differentiation by hand

You should be familiar with the derivatives in Table 1.1.

In this table and in the rules stated below, we have used x as the independent variable. It may be replaced by t or by any other symbol.

In the table, a and n are constants.

Table 1.1

function $f(x)$	derivative $f'(x)$
a	0
$x^n \quad (n \neq 0)$	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
e^{ax}	ae^{ax}
$\ln(ax) \quad (ax > 0)$	$1/x$

You should also be familiar with two important rules which we recap next.

Constant Multiple Rule

Let $f(x)$ be a smooth function and c a constant.

In function notation, if $k(x) = cf(x)$,

$$k'(x) = cf'(x).$$

In Leibniz notation, where $y = f(x)$,

$$\frac{d}{dx}(cy) = c\frac{dy}{dx}.$$

For example, for $f(x) = \sin x$ and $c = 3$, so that $k(x) = 3\sin x$, using function notation,

$$k'(x) = 3f'(x) = 3\cos x,$$

and using Leibniz notation,

$$\frac{d}{dx}(3\sin x) = 3\frac{d}{dx}(\sin x) = 3\cos x.$$

Sum Rule

Let $f(x)$ and $g(x)$ be smooth functions.

In function notation, if $k(x) = f(x) + g(x)$,

$$k'(x) = f'(x) + g'(x).$$

In Leibniz notation, where $u = f(x)$ and $v = g(x)$,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, for $f(x) = \sin x$ and $g(x) = \cos x$, so that $k(x) = \sin x + \cos x$, using function notation,

$$k'(x) = f'(x) + g'(x) = \cos x - \sin x,$$

and using Leibniz notation,

$$\frac{d}{dx}(\sin x + \cos x) = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) = \cos x - \sin x.$$

Now we use Table 1.1 and these rules in some examples.

Example 1.1

Differentiate the following functions.

(a) $k(x) = \cos(4x) - 2x + 3x^5$. (b) $h(t) = \sqrt{t} - 4\ln(t^3)$ $(t > 0)$.

Solution

(a) Using Table 1.1, the Constant Multiple Rule and the Sum Rule,

$$k'(x) = -4\sin(4x) - 2 \times 1 + 3 \times 5x^4 = -4\sin(4x) - 2 + 15x^4.$$

From Table 1.1, the derivative of x is 1, the derivative of x^5 is $5x^4$ and the derivative of $\cos(4x)$ is $-4\sin(4x)$.

(b) First we put the expression $\sqrt{t} - 4\ln(t^3)$ into a form to which we can apply the results in Table 1.1. In power form, $\sqrt{t}$ is $t^{\frac{1}{2}}$ and, by the rules of logarithms, $\ln(t^3)$ is $3\ln t$. Thus, $h(t) = t^{\frac{1}{2}} - 12\ln t$ and, using Table 1.1, the Constant Multiple Rule and the Sum Rule,

$$h'(t) = \tfrac{1}{2}t^{-\frac{1}{2}} - 12 \times t^{-1} = \frac{1}{2\sqrt{t}} - \frac{12}{t}. \quad \blacksquare$$

Exercise 1.1

Differentiate the following functions by hand, using Table 1.1 and the above rules as appropriate.

(a) $f(x) = e^{4x}$, using function notation.

(b) $g(t) = 6t^3$, using function notation.

(c) $s = 4\sin(3t)$, using Leibniz notation.

(d) $y = 7x^4 - 3x^2 + 4$, using Leibniz notation.

(e) $k(x) = 6e^x + \ln(4x)$, using function notation.

If you have difficulty with any of these, you might like to look at MST121 *Chapter C1*. In particular, the exercises there provide further practice.

If $y = f(x)$, the **second derivative** is $f''(x)$. In Leibniz notation, we write

$$\frac{d^2y}{dx^2}, \quad \text{where } \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

All you have to do is differentiate twice.

Exercise 1.2

Find the second derivative of the functions in Exercise 1.1.

1.2 Products, quotients and composites

Product Rule

The derivative of a product of factors is *not* the product of the derivatives of the factors.

Product Rule

Let $f(x)$ and $g(x)$ be smooth functions.

In function notation, if $k(x) = f(x)g(x)$,

$$k'(x) = f'(x)g(x) + f(x)g'(x).$$

In Leibniz notation, where $u = f(x)$ and $v = g(x)$,

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Example 1.2

Differentiate the function $k(x) = x^2 \cos x$.

Solution

Let $f(x) = x^2$ and $g(x) = \cos x$. Then $k(x) = f(x)g(x)$, $f'(x) = 2x$ and $g'(x) = -\sin x$.

Using the Product Rule, we have

$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 2x\cos x + x^2(-\sin x) \\ &= 2x\cos x - x^2\sin x. \end{aligned}$$ ■

You may find the following diagram a help in remembering the Product Rule and also in using it.

$$k(x) = f(x)g(x)$$

differentiate / differentiate

$$k'(x) = f'(x)g(x) + f(x)g'(x)$$

The outer arrows indicate differentiation and the inner ones leave things as they are.

For the previous example, we have the following diagram.

$$k(x) = x^2 \cos x$$

differentiate / differentiate

$$\begin{aligned} k'(x) &= 2x \cos x + x^2(-\sin x) \\ &= 2x \cos x - x^2 \sin x \end{aligned}$$

You may find that, with practice, you can write down the derivative of a product without having to declare explicitly the individual functions in the product and their derivatives.

Exercise 1.3

Differentiate the following functions.

(a) $k(x) = x \cos x$. (b) $k(x) = \sin x \cos x$.

(c) $k(t) = \sin(3t)\cos(4t)$. (d) $k(t) = (3t^2 + 2t + 1)e^t$.

(e) $y = e^x \sin(2x)$. (f) $y = (x^2 + 2)\cos(2x)$.

(g) $f(x) = \sin^2 x$. (h) $f(x) = 2x^2 \sin x + (x^2 + 1)\cos x$

Quotient Rule

Quotient Rule

Let $f(x)$ and $g(x)$ be smooth functions, with $g(x) \neq 0$.

In function notation, if $k(x) = f(x)/g(x)$,

$$k'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

The term $(g(x))^2$ denotes the product $g(x) \times g(x)$.

In Leibniz notation, where $u = f(x)$ and $v = g(x)$,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left(\frac{du}{dx}v - u\frac{dv}{dx}\right) \Big/ v^2.$$

Example 1.3

Differentiate the function $k(x) = \dfrac{x^3 + 2x + 1}{x^2 + 3}$.

Solution

Let $f(x) = x^3 + 2x + 1$ and $g(x) = x^2 + 3$. Then $k(x) = f(x)/g(x)$, $f'(x) = 3x^2 + 2$ and $g'(x) = 2x$.

Using the Quotient Rule, we have

$$\begin{aligned} k'(x) &= \frac{(3x^2 + 2)(x^2 + 3) - (x^3 + 2x + 1)2x}{(x^2 + 3)^2} \\ &= \frac{x^4 + 7x^2 - 2x + 6}{(x^2 + 3)^2}. \quad \blacksquare \end{aligned}$$

When you are used to using the rule, you should be able to write down the last but one line without going through the intermediate step of writing down $f(x)$, $g(x)$, $k(x)$, $f'(x)$ and $g'(x)$ each time.

If you are faced with differentiating a quotient written in symbols different from those in the statement of the rule, you may find it helpful to recast the rule in terms of those symbols. Alternatively, you could use the rule in words:

$$\text{expression} = \frac{\text{top}}{\text{bottom}},$$

$$\text{derivative} = \frac{(\text{derivative of top})(\text{bottom}) \; - \; (\text{top})(\text{derivative of bottom})}{(\text{bottom})^2},$$

that is,

$$\text{derivative} = \frac{(\text{top}')(\text{bottom}) \; - \; (\text{top})(\text{bottom}')}{(\text{bottom})^2}.$$

Example 1.4

Differentiate the function $f(x) = \dfrac{2\cos x}{x} \quad (x \in \mathbb{R},\, x \neq 0)$.

$\mathbb{R}$ denotes the set of real numbers.

Solution

Let $g(x) = 2\cos x$ and $k(x) = x$. Then $f(x) = g(x)/k(x)$, $g'(x) = -2\sin x$ and $k'(x) = 1$.

Using the Quotient Rule, we have

$$f'(x) = \frac{g'(x)k(x) - g(x)k'(x)}{(k(x))^2} = \frac{-2x\sin x - 2\cos x}{x^2}. \quad \blacksquare$$

Exercise 1.4

Differentiate the following functions.

(a) $k(x) = \dfrac{\sin x}{x} \quad (x \in \mathbb{R},\, x \neq 0)$. (b) $k(x) = \dfrac{2x}{1+x} \quad (x \in \mathbb{R},\, x \neq -1)$.

Each of the given domains excludes values at which the denominator is zero.

(c) $k(x) = \dfrac{\sin x}{\cos x} \quad (x \in (-\frac{\pi}{2}, \frac{\pi}{2}))$. (d) $k(t) = \dfrac{t}{\sin t} \quad (t \in (0, \pi))$.

(e) $k(x) = \dfrac{3x^2 - 1}{1 - x} \quad (x \in \mathbb{R},\, x \neq 1)$. (f) $y = \dfrac{\cos x}{2x^2} \quad (x \neq 0)$.

(g) $s = \dfrac{e^{2t}}{\sin(3t)} \quad (t \in (0,\, \frac{\pi}{3}))$.

Trigonometric functions

The trigonometric functions tangent (tan), secant (sec), cosecant (cosec) and cotangent (cot) are defined in terms of the sine (sin) and cosine (cos) functions as:

$$\tan x = \frac{\sin x}{\cos x} \quad (\cos x \neq 0),$$

$$\sec x = \frac{1}{\cos x} \quad (\cos x \neq 0),$$

$$\operatorname{cosec} x = \frac{1}{\sin x} \quad (\sin x \neq 0),$$

$$\cot x = \frac{1}{\tan x} \quad (\tan x \neq 0).$$

We can use the Quotient Rule to find derivatives of these functions. You have already found the derivative of $\tan x$ in Exercise 1.4(c). The answer given is $1/\cos^2 x$, which is $\sec^2 x$.

Exercise 1.5

Using the Quotient Rule, find the derivative of the following functions.

(a) $k(x) = \sec x$. (b) $k(x) = \operatorname{cosec} x$. (c) $k(x) = \cot x$.

Now we can extend our table of derivatives, as in Table 1.2.

Table 1.2

function $f(x)$	derivative $f'(x)$
a	0
$x^n \quad (n \neq 0)$	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan x$	$\sec^2 x$
$\sec x$	$\tan x \sec x$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^{ax}	ae^{ax}
$\ln(ax) \quad (ax > 0)$	$1/x$

Composite Rule

A complicated function can often be broken down into a succession of simpler functions. For example, the function

$$k(x) = e^{x^2} \quad (x \in \mathbb{R})$$

is formed by first applying the function $g(x) = x^2$ and then applying the exponential function $f(x) = e^x$ to the result. In general, given two functions $f(x)$ and $g(x)$, their composition is written

$$f(g(x)).$$

The order in which the functions are applied is important, as $f(g(x)) = e^{x^2}$ is different from $g(f(x)) = (e^x)^2 = e^{2x}$. The Composite Rule gives a way to determine the derived function for such functions.

Composite Rule

For smooth functions $f(x)$ and $g(x)$, if $k(x) = f(g(x))$, then

$$k'(x) = f'(g(x)) \times g'(x).$$

Example 1.5

Differentiate $k(x) = e^{x^2}$, using the Composite Rule.

Solution

Let $f(x) = e^x$ and $g(x) = x^2$, so that $k(x) = f(g(x))$. Since $f'(x) = e^x$ and $g(x) = x^2$, $f'(g(x)) = e^{g(x)} = e^{x^2}$. Also, $g'(x) = 2x$.

Using the Composite Rule,

$$\begin{aligned} k'(x) &= f'(g(x)) \times g'(x) \\ &= e^{x^2} \times 2x \\ &= 2xe^{x^2}. \quad \blacksquare \end{aligned}$$

In Leibniz notation, the rule is rather easier to remember. Let $u = g(x)$ (the inner function) and $y = f(g(x))$, so $y = f(u)$. Then

$$\frac{dy}{du} = f'(u) = f'(g(x)), \tag{1.1}$$

$$\frac{du}{dx} = g'(x). \tag{1.2}$$

Now, by the Composite Rule,

$$\frac{dy}{dx} = f'(g(x)) \times g'(x).$$

Equations (1.1) and (1.2) enable us to express the Composite Rule in Leibniz notation as

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This form is often known as the **Chain Rule** and you may find it easier to remember because it looks as if the two terms 'du' are being cancelled.

Mathematically this is not correct, since the symbol du on its own does not have a meaning. Nevertheless, this is a useful mnemonic.

Chain Rule

The derived function of $y = f(g(x))$, where $u = g(x)$ and $y = f(u)$, is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Example 1.6

Differentiate the function $k(x) = \sin(x^2)$.

Solution

The outer function is sine and the inner function is the squaring function. Let $g(x) = x^2$ and $f(x) = \sin x$, so $k(x) = f(g(x))$. Then $f'(x) = \cos x$, so $f'(g(x)) = \cos(g(x)) = \cos(x^2)$. Also, $g'(x) = 2x$.

Using the Composite Rule, we have

$$\begin{aligned} k'(x) &= f'(g(x)) \times g'(x) \\ &= \cos(x^2) \times 2x \\ &= 2x\cos(x^2). \end{aligned}$$

Here is an alternative solution using the Chain Rule. Let $y = \sin(x^2)$ and $u = x^2$, so $y = \sin u$. Then $dy/du = \cos u$ and $du/dx = 2x$.

Using the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times 2x \\ &= \cos(x^2) \times 2x \\ &= 2x\cos(x^2). \quad \blacksquare\end{aligned}$$

Example 1.7

Differentiate $k(x) = \sin^3 x$.

Solution

First we need to decide which is the inner function and which is the outer function. Remembering that $\sin^3 x = (\sin x)^3$, we see that the outer function is the cubing function and the inner function is sine.

Let $g(x) = \sin x$ and $f(x) = x^3$, so $k(x) = f(g(x))$. Then $f'(x) = 3x^2$, so $f'(g(x)) = 3(g(x))^2 = 3(\sin x)^2 = 3\sin^2 x$. Also $g'(x) = \cos x$.

Using the Composite Rule, we have

$$\begin{aligned}k'(x) &= f'(g(x))g'(x) \\ &= 3\sin^2 x \times \cos x \\ &= 3\sin^2 x \cos x.\end{aligned}$$

An alternative solution using the Chain Rule is as follows.

Let $y = \sin^3 x$, $u = \sin x$, so $y = u^3$. Then $dy/du = 3u^2$ and $du/dx = \cos x$.

Using the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3u^2 \times \cos x \\ &= 3\sin^2 x \times \cos x \\ &= 3\sin^2 x \cos x. \quad \blacksquare\end{aligned}$$

As you become more familiar with applying the Composite Rule and the Chain Rule, you might find that you can just 'write down' the derivative of a simple composite function, no matter which form of the rule you are using.

Exercise 1.6

Verify that the derivatives of the following are as given in Table 1.2. You may assume that the derivative of $\sin x$ is $\cos x$, of $\cos x$ is $-\sin x$, of e^x is e^x and of $\ln x$ $(x > 0)$ is $1/x$.

(a) $\sin(ax)$. (b) $\cos(ax)$. (c) e^{ax}. (d) $\ln(ax)$ $(ax > 0)$.

Exercise 1.7

Differentiate the following functions.

(a) $\cos(x^4)$. (b) $e^{\operatorname{cosec} x}$ $(\sin x \neq 0)$. (c) $\cos^2 x$. (d) $\ln(1 + x^6)$.

(e) $(t^3 + 2)^5$. (f) $\cot(t^2 - 3t + 1)$ $(\sin(t^2 - 3t + 1) \neq 0)$.

(g) $\sqrt{x^3 + 4x + 3}$ $(x > 0)$.

For (a), (c), (d) and (e), the domain is all of $\mathbb{R}$. In (b), (f) and (g), the domains are restricted as given.

Differentiating more complicated expressions

You should now be able to use any combination of the rules for differentiation.

Example 1.8

Differentiate the function

$$q(x) = \frac{\cos(x^2 + 2x - 1)}{x} \quad (x \neq 0).$$

Solution

This expression is a quotient, so we need to use the Quotient Rule. The 'top' function is a composite function, in which the inner function is made up from functions whose derivatives are given in Table 1.2.

Let $h(x) = \cos(x^2 + 2x - 1)$ and $k(x) = x$, so $q(x) = h(x)/k(x)$. Then, by the Quotient Rule,

$$q'(x) = \frac{h'(x)k(x) - h(x)k'(x)}{(k(x))^2}. \tag{1.3}$$

We shall find the derivative of the composite function $h(x)$ as a subproblem.

Let $h(x) = f(g(x))$, where $f(x) = \cos x$ and $g(x) = x^2 + 2x - 1$. Since $f'(x) = -\sin x$, we have $f'(g(x)) = -\sin(g(x)) = -\sin(x^2 + 2x - 1)$. Also $g'(x) = 2x + 2$.

By the Composite Rule,

$$\begin{aligned} h'(x) &= f'(g(x)) \times g'(x) \\ &= -\sin(x^2 + 2x - 1) \times (2x + 2) \\ &= -(2x + 2)\sin(x^2 + 2x - 1). \end{aligned}$$

Returning to Equation (1.3),

$$\begin{aligned} q'(x) &= \frac{h'(x)k(x) - h(x)k'(x)}{(k(x))^2} \\ &= \frac{\big(-(2x + 2)\sin(x^2 + 2x - 1) \times x\big) - \big(\cos(x^2 + 2x - 1) \times 1\big)}{x^2} \\ &= \frac{-x(2x + 2)\sin(x^2 + 2x - 1) - \cos(x^2 + 2x - 1)}{x^2}. \end{aligned}$$ ■

Example 1.9

Differentiate the function $f(x) = x^4 \sin x \cos x$.

Solution

Here we have a triple product. You may find it easier to treat it as two problems, as above. Let us differentiate $x^4 \sin x$ as a subproblem and then differentiate the product of $x^4 \sin x$ with $\cos x$.

Let $g(x) = x^4 \sin x$, $h(x) = x^4$ and $k(x) = \sin x$, so $g(x) = h(x) \times k(x)$. Then, by the Product Rule, $g'(x) = h'(x)k(x) + h(x)k'(x)$. The process is illustrated below.

$$g(x) = x^4 \sin x$$

differentiate / differentiate

$$g'(x) = 4x^3 \sin x + x^4 \cos x$$

So $g'(x) = 4x^3 \sin x + x^4 \cos x$.

Next we apply the Product Rule to $f(x) = (x^4 \sin x)(\cos x)$, as illustrated below.

$$f(x) = (x^4 \sin x)(\cos x)$$

differentiate differentiate

$$f'(x) = (4x^3 \sin x + x^4 \cos x)\cos x + (x^4 \sin x)(-\sin x)$$

So $f'(x) = 4x^3 \sin x \cos x + x^4 \cos^2 x - x^4 \sin^2 x$. ■

Exercise 1.8

Differentiate the following functions.

(a) $f(x) = x \ln(1 + x + x^2) \quad (x \in \mathbb{R})$.

(b) $f(x) = x^3 \tan x \ln x \quad (x \in (0, \frac{\pi}{2}))$.

(c) $y = \cos((x + 4)\sec x) \quad (\cos x \neq 0)$.

(d) $y = \sin(\sin(\sin x)) \quad (x \in \mathbb{R})$.

The most difficult of these is probably (d), but none of them is very easy. You will *not* be expected to differentiate by hand more complicated functions than these.

Some practical examples

We end this subsection by asking you to apply the rules of differentiation that we have considered so far in this section to some practical examples.

Some of these examples concern the motion of an object in a straight line. If s is the displacement of the object from some fixed point on that line at some time t, then

$$\frac{ds}{dt} \quad \text{and} \quad \frac{d^2s}{dt^2}$$

give the velocity and the acceleration of the object, respectively.

Other examples here concern the use of the Second Derivative Test. This test can be used to decide whether a stationary point of a function is a local minimum or a local maximum. A stationary point of a function $f(x)$ is a point where the graph of $f(x)$ has zero gradient and so where $f'(x) = 0$. If a is a stationary point of $f(x)$ and $f''(a)$ is negative, by the Second Derivative Test, a is a local maximum. If a is a stationary point of $f(x)$ and $f''(a)$ is positive, by the Second Derivative Test, a is a local minimum.

See MST121 *Chapter C1*.

Exercise 1.9

The distance s (measured in metres) of an oscillating object from a fixed point is given by $s = e^{2t}\cos(\pi t)$, where t is the time since the start of the motion and is measured in seconds. Find the velocity and acceleration of the object. What are their values after one second from the start of the motion?

Exercise 1.10

Show that the function $f(x) = xe^{-x}$ has a local maximum at the point $(1, e^{-1})$.

Do not forget to factorize the first derivative so that you can find the values where it is zero.

Exercise 1.11

If $y = x^2/(1 - x)$, find the coordinates of the points where the tangent to the graph of this function is horizontal.

Exercise 1.12

Find the stationary point on the curve $y = e^{-x^2}$ and show that it is a local maximum.

1.3 Inverse functions

Logarithm function

So far we have used the fact that

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

without really knowing why this is true. To see how to differentiate the logarithm function, we can use the Composite Rule in a rather cunning way. We make use of the fact that the exponential and logarithm functions are *inverses* of one another, a fact whose meaning we now revise. If a function f is one–one, then it has an *inverse* function f^{-1} that 'undoes' the effect of f and vice-versa, so that

The notation f^{-1} does *not* mean $1/f$.

$$f^{-1}(f(x)) = x.$$

The exponential function e^x is a one–one function from the real numbers $\mathbb{R}$ to the positive real numbers $\mathbb{R}^+$ (e^x is never negative or zero). Its inverse is the logarithm function since,

$$\ln(e^x) = x$$

for each x in $\mathbb{R}$, and

$$e^{\ln y} = y \tag{1.4}$$

for each y in $\mathbb{R}^+$.

Now let us use the Composite Rule and Equation (1.4) to compute the derivative of the logarithm function. By differentiating both sides of Equation (1.4) with respect to y,

In the case of the exponential function, $f(x) = e^x$ and $f^{-1}(x) = \ln x$, whereas $1/f(x) = 1/e^x = e^{-x}$.

$$e^{\ln y} \times \ln' y = 1. \tag{1.5}$$

But since ln and exp are inverses, $e^{\ln y} = y$ and Equation (1.5) becomes

$$y \times \ln' y = 1.$$

Dividing by y (y is positive and therefore non-zero), we obtain

$$\ln' y = \frac{1}{y}$$

which is the result we were after.

Inverse trigonometric functions

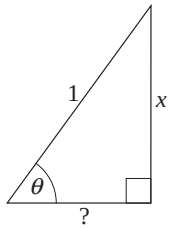
You will not be expected to differentiate the inverse trigonometric functions by hand in MST209. However, for completeness, we show the method.

On the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the sine function is one–one and maps onto $[-1, 1]$. The inverse function (which can be thought of as 'the angle whose sine is ...') is denoted by arcsin, so we have

Looked at another way, $\cos(\arcsin x)$ is 'the cosine of the angle whose sine is x', as illustrated below.

$$\sin(\arcsin x) = x, \tag{1.6}$$

for each x in $[-1, 1]$.



Differentiating Equation (1.6) using the Composite Rule gives

$$\cos(\arcsin x)\frac{d}{dx}(\arcsin x) = 1.$$

Since $\cos^2\theta + \sin^2\theta = 1$ for any θ, the cosine of an angle whose sine is x is $\sqrt{1-x^2}$, so

By Pythagoras's Theorem, the third side of the triangle is of length $\sqrt{1-x^2}$, so $\cos\theta = \dfrac{1}{\sqrt{1-x^2}}$.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (x \in (-1, 1)).$$

Similar results for the inverses of the cosine and tangent functions over suitable domains enable us to complete Table 1.3.

Table 1.3

function $f(x)$	derivative $f'(x)$
a	0
$x^n \quad (n \neq 0)$	nx^{n-1}
$\sin(ax)$	$a\cos(ax)$
$\cos(ax)$	$-a\sin(ax)$
$\tan x$	$\sec^2 x$
$\sec x$	$\tan x \sec x$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^{ax}	ae^{ax}
$\ln(ax) \quad (ax > 0)$	$1/x$
$\arcsin x \quad (-1 < x < 1)$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x \quad (-1 < x < 1)$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Exercise 1.13

Differentiate the following.

(a) $y = \arcsin(x^3)$. (b) $y = \ln(-x) \quad (x < 0)$.

1.4 Approximating functions using polynomials

To obtain values of e^x, $\sin x$, $\cos x$ and $\tan x$ for a particular value of x, it is usual to turn to a calculator or computer, since there are no simple arithmetic operations that will yield the exact values. How does the calculator or computer calculate a value? Is it possible to obtain acceptable approximations for these functions at a particular value of x by simple arithmetic?

The answer is that there are several methods and one of these involves approximating the function by a polynomial.

We begin by looking at the simplest type in which the polynomial is linear. Then we shall look at quadratic approximations before moving to higher-order polynomial approximations.

Tangent approximation

First we shall derive a straight line (linear) approximation to a curve at a particular point.

Consider the definition of the derivative of a function $f(x)$ at $x = a$,

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right). \tag{1.7}$$

In the limit, this represents the gradient of the curve at $x = a$, as Figure 1.2 illustrates. For small non-zero values of h, the quotient on the right-hand side of Equation (1.7) will give a value close to that of $f'(a)$, so we can write

$$f'(a) \simeq \frac{f(a+h) - f(a)}{h}.$$

The curly equals sign means 'approximately equals'.

This may be rearranged to give

$$f'(a)h \simeq f(a+h) - f(a).$$

Writing $x = a + h$, substituting $a + h = x$ and $h = x - a$ from this, and rearranging again, gives

$$f(x) \simeq f(a) + (x-a)f'(a).$$

The right-hand side is a simple approximation to $f(x)$ near $x = a$.

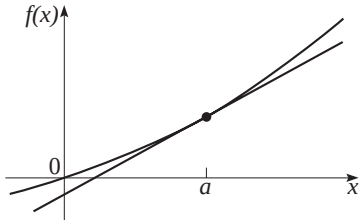


Figure 1.2

> ***Definition***
>
> The **tangent approximation** to the function $f(x)$ near $x = a$ is a linear polynomial $p(x)$, where
>
> $$p(x) = f(a) + (x-a)f'(a).$$

Notice that $p(a) = f(a)$ and $p'(x) = f'(a)$, so the approximation and the function have the same value and the same derivative at $x = a$.

Example 1.10

Find the tangent approximation to e^x near $x = 0$.

Solution

In this case, $f(x) = e^x$ and $a = 0$, so

$$\begin{aligned} p(x) &= f(0) + (x-0)f'(0) \\ &= e^0 + (x-0)e^0 \\ &= 1 + (x-0)1 \\ &= 1 + x. \end{aligned}$$

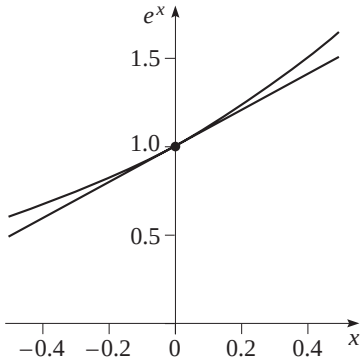


Figure 1.3

Figure 1.3 shows this approximation near $x = 0$. The approximation looks quite reasonable for $-0.2 < x < 0.2$. However, the approximation becomes less accurate the further the value of x from 0. ■

Example 1.11

Find the tangent approximation to $f(x) = x + \ln x$ near $x = 1$.

Solution

We have $f(x) = x + \ln x$ and $a = 1$, so

$$f(1) = 1 + \ln 1 = 1 + 0 = 1.$$

Differentiating $f(x)$ gives

$$f'(x) = 1 + \frac{1}{x},$$

so that

$$f'(1) = 1 + 1 = 2.$$

Hence

$$p(x) = 1 + (x-1)2$$
$$= 2x - 1.$$

Figure 1.4 shows this tangent approximation near $x = 1$. In this case, the figure shows that the tangent approximation is reasonable for $0.7 < x < 1.2$. However, you can again see that the accuracy reduces the further the value of x from 1. ■

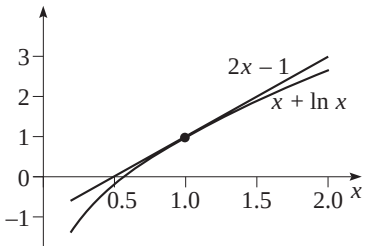


Figure 1.4

Exercise 1.14

(a) Find the tangent approximation to $f(x) = \sin x$ near $x = 0$.

(b) Find the tangent approximation to $f(x) = \cos x$ near $x = 0$.

(c) Find the tangent approximation to $f(x) = e^x$ near $x = 1$.

(d) Find the tangent approximation to $f(x) = x^3 + 2x - 5$ near $x = 1$.

Quadratic approximation

A function and its tangent approximation have the same value at the point where the approximation is made ($p(a) = f(a)$) and the same derivative at this point ($p'(a) = f'(a)$), by the definition of the tangent approximation. Now suppose we try to increase the accuracy of the approximation by using a quadratic instead of a linear function and adding a quadratic term $(x-a)^2$. In other words, we shall try an approximation of the form

$$p_2(x) = f(a) + (x-a)f'(a) + (x-a)^2 c. \tag{1.8}$$

But what value should the coefficient c take?

We choose c so that $p_2(x)$ and $f(x)$ have the same *second* derivative at a. We can do this by differentiating both sides of Equation (1.8) twice, treating c as a constant, to obtain

$$p_2'(x) = f'(a) + 2(x-a)c,$$
$$p_2''(x) = 2c.$$

If $f(x)$ and $p_2(x)$ are to have the same second derivative at a, then

$$f''(a) = p_2''(a) = 2c,$$

so that

$$c = \tfrac{1}{2} f''(a).$$

Thus the *quadratic approximation* for $f(x)$ near $x = a$ is

$$f(x) \simeq f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a).$$

Definition

The **quadratic approximation** to the function $f(x)$ near $x = a$ is a quadratic polynomial $p_2(x)$, where

$$p_2(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a). \tag{1.9}$$

Example 1.12

Find the quadratic approximation for e^x near $x = 0$.

Solution

The first two terms were found in Example 1.10. To find the third term, we use the fact that $f''(x) = e^x$, so $f''(0) = 1$ and, from Equation (1.9),

$$p_2(x) = 1 + x + \tfrac{1}{2}x^2.$$

Figure 1.5 contrasts this quadratic approximation with the tangent approximation derived in Example 1.10.

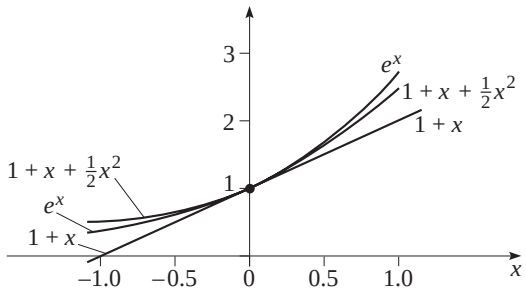


Figure 1.5

The quadratic approximation is clearly an improvement over the tangent approximation near $x = 0$. ■

Exercise 1.15

(a) Find the quadratic approximation to $f(x) = \cos x$ near $x = 0$.

(b) Find the quadratic approximation to $f(x) = \sin x$ near $x = 0$.

(c) Find the quadratic approximation to $f(x) = \ln(1 + x)$ near $x = 0$.

(d) Find the quadratic approximation to $f(x) = \sqrt{1 + x}$ near $x = 0.2$.

Taylor polynomials

In obtaining a linear approximation $p_1(x)$ for $f(x)$ near $x = a$, we imposed the conditions $p(a) = f(a)$ and $p'(a) = f'(a)$. For the quadratic approximation $p_2(x)$, these conditions together with the condition $p''(a) = f''(a)$ were imposed. Such polynomials are called **Taylor polynomials** and, when used as approximations to functions, are called **Taylor approximations**.

The improvement in accuracy in moving from a linear to a quadratic approximation suggests that further improvements might be possible by using higher powers of $(x - a)$. The additional condition in using a cubic, quartic, ... approximation would be $p'''(a) = f'''(a)$, $p^{(4)}(a) = f^{(4)}(a)$, ... Provided we know the values of the higher derivatives of the function at $x = a$, the same method would work. It turns out that many suitably smooth functions can be approximated as closely as required, provided that enough terms are added. There is a pattern to the terms that is easy to remember.

By 'suitably smooth', we mean that sufficient of its derivatives exist near $x = a$.

Definition

For a function $f(x)$ that has n continuous derivatives near $x = a$, the **Taylor polynomial of degree n about $x = a$** or **nth-order Taylor polynomial about $x = a$** is

$$p_n(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2!}(x-a)^2 f''(a) + \tfrac{1}{3!}(x-a)^3 f'''(a) + \cdots + \tfrac{1}{n!}(x-a)^n f^{(n)}(a). \quad (1.10)$$

A *continuous* function is one whose graph can be drawn without lifting the pencil from the paper.

The notation $n!$ means $n(n-1)(n-2)\ldots 1$.

The Taylor polynomial of degree 1, a linear polynomial, for the function $f(x)$ about $x = a$ is

$$p_1(x) = f(a) + (x-a)f'(a)$$

and is the same as the tangent approximation.

The Taylor polynomial of degree 2, a quadratic polynomial, for the function $f(x)$ about $x = a$ is

$$p_2(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a)$$

and is the same as the quadratic approximation.

Example 1.13

Show that

$$p(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5$$

is the Taylor polynomial of degree 5 for e^x about $x = 0$.

Solution

We have

$$f(x) = e^x, \quad f^{(n)}(x) = e^x \quad (n = 1, \ldots, 5),$$
$$f(0) = 1, \quad f^{(n)}(0) = 1 \quad (n = 1, \ldots, 5).$$

Hence, using Equation (1.10) with $a = 0$,

$$p_5(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \tfrac{1}{120}x^5. \quad \blacksquare$$

Example 1.14

Find the Taylor polynomial of degree 3 about $x = \frac{\pi}{6}$ for the function $f(x) = \sin x$.

Solution

We have

$$f(x) = \sin x, \quad f(\tfrac{\pi}{6}) = \tfrac{1}{2},$$
$$f'(x) = \cos x, \quad f'(\tfrac{\pi}{6}) = \tfrac{\sqrt{3}}{2},$$
$$f''(x) = -\sin x, \quad f''(\tfrac{\pi}{6}) = -\tfrac{1}{2},$$
$$f'''(x) = -\cos x, \quad f'''(\tfrac{\pi}{6}) = -\tfrac{\sqrt{3}}{2}.$$

Substituting these values into Equation (1.10) with $a = \frac{\pi}{6}$, we obtain

$$p_3(x) = f(\tfrac{\pi}{6}) + (x-\tfrac{\pi}{6})f'(\tfrac{\pi}{6}) + \tfrac{1}{2!}(x-\tfrac{\pi}{6})^2 f''(\tfrac{\pi}{6}) + \tfrac{1}{3!}(x-\tfrac{\pi}{6})^3 f'''(\tfrac{\pi}{6})$$
$$= \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}(x-\tfrac{\pi}{6}) - \tfrac{1}{4}(x-\tfrac{\pi}{6})^2 - \tfrac{\sqrt{3}}{12}(x-\tfrac{\pi}{6})^3. \quad \blacksquare$$

As well as giving approximate algebraic expressions for functions, Taylor polynomials may be used to find approximate values for functions at particular values of their arguments.

Example 1.15

Find the Taylor polynomial of degree 5 for $\sin x$ about $x = 0$. Hence obtain a Taylor approximation for the value of $\sin 0.2$, correct to three decimal places.

Solution

For $f(x) = \sin x$, we have

$$f(0) = \sin 0 = 0,$$
$$f'(0) = \cos 0 = 1,$$
$$f''(0) = -\sin 0 = 0,$$
$$f'''(0) = -\cos 0 = -1,$$
$$f^{(4)}(0) = \sin 0 = 0,$$
$$f^{(5)}(0) = \cos 0 = 1.$$

Substituting these values into Equation (1.10) with $a = 0$, we obtain

$$\begin{aligned}\sin x &\simeq x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 \\ &= x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5.\end{aligned}$$

From this we can deduce that the Taylor polynomials of degrees 1 to 4 about $x = 0$ are

$$p_1(x) = p_2(x) = x,$$
$$p_3(x) = p_4(x) = x - \tfrac{1}{6}x^3.$$

Now we look for two successive non-zero approximations to $\sin x$ at $x = 0.2$, the values of which agree to $3 + 2 = 5$ decimal places. The results are

$$p_1(0.2) = 0.2,$$
$$p_3(0.2) = p_1(0.2) - \tfrac{1}{6}(0.2)^3 = 0.198\,67,$$
$$p_5(0.2) = p_3(0.2) + \tfrac{1}{120}(0.2)^5 = 0.198\,67.$$

So the required approximation is

$$\sin 0.2 = 0.199 \quad \text{(to three decimal places)}. \quad \blacksquare$$

To be reasonably confident that convergence to n decimal places has occurred, we look for two successive values that agree to $n + 2$ decimal places. An alternative method is to look for four values that agree to n decimal places. However, neither method *guarantees* that convergence has been achieved.

Exercise 1.16

Find the cubic Taylor polynomial about $x = 0$ for $\ln(1 + x)$.

Exercise 1.17

(a) Show that the Taylor polynomial of degree 5 about $x = 0$ for the function $f(x) = \sqrt{1 + x}$ is

$$p_5(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5.$$

(b) By expressing 10 as $9(1 + \frac{1}{9})$, use (a) to evaluate $\sqrt{10}$ to an accuracy of four decimal places without using the square root key on your calculator.

You may wonder what would happen if this process were continued indefinitely: this involves a mathematical leap — towards infinity! It turns out that for many functions it makes sense to talk about an 'infinite Taylor polynomial', although the technical term is **Taylor series**. A Taylor series gives an *exact* algebraic formula for the function. It is not an approximation, as are the finite Taylor polynomials. For many of the functions we have looked at, the Taylor series can be written down directly as an extension

of the Taylor polynomial of degree n. For example, the Taylor series of the exponential, sine and cosine functions are given by the following.

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots + \tfrac{1}{n!}x^n + \cdots$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + (-1)^{n-1}\tfrac{1}{(2n-1)!}x^{2n-1} + \cdots$$

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots + (-1)^n\tfrac{1}{(2n)!}x^{2n} + \cdots$$

We shall see these again in Section 3.

End-of-section Exercises

Exercise 1.18

Differentiate the following functions, stating the rules you use.

(a) $f(x) = \sin(e^x)$. (b) $f(x) = \cos^2(3x)$.

(c) $g(y) = \dfrac{\ln(\cos y)}{y}$ $(0 < y < \frac{\pi}{2})$. (d) $k(t) = \sin(t^2)\cos t$.

(e) $y = 2\sin(t+1) - t\cos(t^2 - 1)$.

Exercise 1.19

The area of the surface of a pyramid with a square base is $18\,\text{cm}^2$. If the lengths of the sides of the square base are x, the volume of the pyramid is $x\sqrt{9 - x^2}$. Find the value of x that gives the greatest volume and show that the greatest volume is $4.5\,\text{cm}^3$.

This exercise is quite challenging, so do not spend too long on it if you find it difficult.

Exercise 1.20

Find the linear, quadratic and cubic Taylor polynomials for the following functions about $x = 1$.

(a) $f(x) = e^{x^2}$. (b) $f(x) = \dfrac{1}{2 - x}$. (c) $f(x) = \ln x$.

2 Integration

One main objective of this section is to present techniques for finding integrals. We shall also use the work in Section 1, since there is a close relationship between differentiation and integration. Subsection 2.1 begins by revising this relationship and recaps some basic methods of integration. The definite integral is discussed and interpreted in terms of areas bounded by graphs. Then, in Subsections 2.2 and 2.3, we look at two important techniques to extend the range of functions that can be integrated by hand. Subsection 2.2 deals with *integration by parts*, which is a technique based on the Product Rule for differentiation. Subsection 2.3 presents *integration by substitution*, a technique which uses the Composite Rule for differentiation.

If you have just finished MST121, you may be able to study Subsection 2.1 very quickly.

2.1 Differential equations and integration

Indefinite integral

We start by observing that integration is the 'opposite' of differentiation and look at an example.

The relation between the function $f(x) = 5x^2 - 6\cos(4x)$ and its indefinite integral $F(x)$ is given by the equation

$$\frac{d}{dx}(F(x)) = 5x^2 - 6\cos(4x).$$

Any such equation involving a derivative of an unknown function is called a **differential equation**. Differential equations may arise in modelling, for example, in situations involving the motion of an object in a single direction. Functions for the position $s(t)$, velocity $v(t)$ and acceleration $a(t)$ at time t are related by the differential equations

$$\dot{v}(t) = a(t), \tag{2.1}$$

$$\dot{s}(t) = v(t). \tag{2.2}$$

Remember that $\dot{v}$ is Newton's notation for dv/dt and is often used in mechanics.

If the acceleration $a(t)$ is known, Equation (2.1) can be solved to find an expression for the velocity $v(t)$. Equation (2.2) can then be solved to find an expression for the position $s(t)$. In each case, we have a differential equation of the general form

$$F'(t) = f(t),$$

where $f(t)$ is a known function and we wish to find $F(t)$.

There are various notations in which such a differential equation might be written. The notation above and the Leibniz notation

$$\frac{dy}{dt} = f(t),$$

where $y = F(t)$, incorporate explicitly an independent variable t. Any letter may be used as the independent variable. For general discussions in this section we shall settle on x, writing the equation as

$$F'(x) = f(x).$$

Note that we use F' to denote the derivative when the independent variable is not t.

The most general form of the function $F(x)$ that satisfies this differential equation is called the **general solution** of the equation. Thus the differential equation

$$F'(x) = x^2 + \cos x \tag{2.3}$$

has general solution

$$F(x) = \tfrac{1}{3}x^3 + \sin x + c,$$

where c is an arbitrary constant.

This is true because:

- the derivative of $F(x) = \frac{1}{3}x^3 + \sin x + c$ is $f(x) = x^2 + \cos x$, for any value of c;
- there is no solution to Equation (2.3) other than those described by the expression $\frac{1}{3}x^3 + \sin x + c$.

The first of these points can be checked by differentiating the formula given for $F(x)$. The second point is not obvious and we shall indicate later why it is true.

In saying above that c is an 'arbitrary constant', we mean that a valid solution of the differential equation is obtained no matter what value c takes. It is only when c is unassigned that the corresponding expression for $F(x)$ represents the *general* (rather than a *particular*) solution.

The general solution is an expression that describes the family of all possible particular solutions.

The check suggested above can be written as

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3 + \sin x + c\right) = x^2 + \cos x,$$

showing explicitly that a process (differentiation, indicated by d/dx) is applied to $F(x) = \frac{1}{3}x^3 + \sin x + c$ in order to obtain $f(x) = x^2 + \cos x$. To solve the equation $F'(x) = x^2 + \cos x$, we need to *undo* this process. We write the outcome of this undoing using the *indefinite integral* notation

$$\int \left(x^2 + \cos x\right) dx = \tfrac{1}{3}x^3 + \sin x + c.$$

Generalizing from this example, the general solution of the differential equation

$$F'(x) = f(x), \tag{2.4}$$

where $f(x)$ is a known function, is

$$F(x) = \int f(x)\, dx. \tag{2.5}$$

The right-hand side of Equation (2.5) is called the *indefinite integral* of the function $f(x)$ and is obtained by *integrating* ('antidifferentiating') the function.

In an integral, the symbol $\int$ is called the *integral sign*, and the function $f(x)$ being integrated is called the *integrand*. The dx on the end indicates that the integration (the process of obtaining the integral) is to be carried out with respect to the variable x.

Definitions

(a) A **differential equation** is an equation containing a derivative of an unknown function.

(b) The general form of the function $F(x)$ that satisfies the differential equation $F'(x) = f(x)$, where $f(x)$ is a known function, is the **general solution**. The general solution of $F'(x) = f(x)$ is

$$F(x) = \int f(x)dx. \tag{2.5}$$

(c) The right-hand side of Equation (2.5) is the **indefinite integral** of $f(x)$ and is obtained by 'antidifferentiating', known as **integrating**, the **integrand** $f(x)$.

The *differential equation*

$F'(x) = f(x) = x^2 + \cos x,$

has *general solution*

$$f(x) = \int x^2 + \cos x\, dx,$$
$$= \tfrac{1}{3}x + \sin x + c.$$

The *indefinite integral* of $f(x)$ is

$\frac{1}{3}x + \sin x + c$

and the *integrand* is

$x^2 + \cos x.$

Equations (2.4) and (2.5) express the same relationship between $f(x)$ and $F(x)$, but use different notations. How does this change of notation take us further forward in finding the general solution for a differential equation of the form of Equation (2.4)?

Certainly, much remains to be done in order to have a workable approach. However, an important first step has been taken. Integration, arising as the 'undoing' of differentiation, is to be regarded as a process in its own right, operating on a known function to produce further known functions.

With differentiation, we first established the derivatives of some simple functions, and progressively built up the repertoire of functions that could be differentiated by establishing various rules for dealing with combinations of simple functions. The same approach can be adopted for integration, taking the indefinite integrals in Table 2.1 as our initial repertoire.

Table 2.1

$f(x)$	$\int f(x)\,dx$
a	$ax + c$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + c$
$1/x \quad (x > 0)$	$\ln x + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + c$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + c$

In each case, c is an arbitrary constant.

In the table, n is any constant not equal to -1 and a is any non-zero constant.

Two rules of integration are now stated.

Constant Multiple Rule
If $f(x)$ is a smooth function and k is a constant,

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Sum Rule
If $f(x)$ and $g(x)$ are smooth functions,

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Exercise 2.1

Find the following indefinite integrals.

(a) $\displaystyle\int 5x + x^3\,dx.$ (b) $\displaystyle\int \frac{1+\sqrt{x}}{x}\,dx \quad (x > 0).$

(c) $\displaystyle\int e^{7u}\,du.$ (d) $\displaystyle\int \sin(2t) - 3\cos(3t)\,dt.$

(e) $\displaystyle\int \cos^2 x\,dx,$ given that $\cos(2x) = 2\cos^2 x - 1.$

Integrals of further functions arise from some of your work in Section 1. For example, you showed in Exercise 1.4 (using the Quotient Rule) that

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

It follows from this that

$$\int \sec^2 x\,dx = \tan x + c,$$

where c is an arbitrary constant.

Table 2.2 lists some further examples.

Table 2.2

$f(x)$	$\int f(x)dx$
$\sec^2 x$	$\tan x + c$
$-\operatorname{cosec}^2 x$	$\cot x + c$
$\tan x \sec x$	$\sec x + c$
$-\cot x \operatorname{cosec} x$	$\operatorname{cosec} x + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c \quad (-1 < x < 1)$
$\dfrac{1}{1+x^2}$	$\arctan x + c$

Exercise 2.2

Use Table 2.2 to find the following indefinite integrals.

(a) $\displaystyle\int \frac{1}{1+x^2}\,dx.$ (b) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du \quad (-1 < u < 1).$ (c) $\displaystyle\int \operatorname{cosec}^2 t\,dt.$

From now on we shall draw on the standard integrals in Tables 2.1 and 2.2.

In general, if we can recognize any function as a derivative, as a multiple of a derivative or as a sum of derivatives, then we can find its integral.

Example 2.1

Find the indefinite integral $\displaystyle\int \frac{x}{1+x^2}\,dx.$

Solution

Using the rule for differentiation of a composite function, we have

$$\frac{d}{dx}\left(\ln(1+x^2)\right) = \frac{2x}{1+x^2}.$$

Hence we can find the integral of $x/(1+x^2)$, as

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int \frac{2x}{1+x^2}\,dx \\ &= \tfrac{1}{2}\ln(1+x^2) + c. \quad \blacksquare\end{aligned}$$

Exercise 2.3

(a) Calculate $\displaystyle\frac{d}{dx}\left(\frac{1}{1+x^2}\right).$

(b) Find $\displaystyle\int \frac{x}{(1+x^2)^2}\,dx$, using the result of (a).

We use the term an **integral** of $f(x)$ to denote a *particular* solution of the differential equation $F'(x) = f(x)$. Thus an integral is obtained from the indefinite integral $\int f(x)\,dx$ by specifying a particular value for the arbitrary constant c that features in the expression for the indefinite integral. For example, the differential equation

$$F'(x) = \sec^2 x$$

has the general solution $\tan x + c$, where c is an arbitrary constant. Correspondingly, $\tan x$, $\tan x + 1$ and $\tan x - 3$ are all integrals of $\sec^2 x$.

Definite integral

The indefinite integral of a function $f(x)$ specifies a set of *functions* — the family of functions that are integrals of $f(x)$. Certain modelling applications lead to a related idea — the *definite* integral.

Before going any further, let us clear up one outstanding matter. You have seen that if the function $f(x)$ has an integral $F(x)$, then $F(x) + c$ is another integral. We stated that all integrals arise in this way. To see why this is so, we make the following reasonable assumption.

Assumption
If the derivative of a smooth function $f(x)$ is zero everywhere, then $f(x)$ is a constant function.

This is plausible because, if the derivative is always zero, then, intuitively, the function does not change. This assumption can be proved, but we do not give the proof here.

Now suppose that $F_1(x)$ and $F_2(x)$ are two integrals of $f(x)$. This means that

$$\frac{d}{dx}(F_1(x)) = f(x) = \frac{d}{dx}(F_2(x)),$$

so that

$$\frac{d}{dx}(F_1(x)) - \frac{d}{dx}(F_2(x)) = 0. \tag{2.6}$$

Using the Sum and Constant Multiple rules for differentiation, we can rewrite the left-hand side of Equation (2.6) as

$$\frac{d}{dx}(F_1(x) - F_2(x)),$$

so that $F_1(x) - F_2(x)$ is a function whose derivative is zero everywhere. Therefore it is a constant and we indeed have

$$F_1(x) = F_2(x) + c \tag{2.7}$$

as claimed. Now we can resume the story.

Let a and b be two values in the domain of a function $f(x)$, and suppose that $F_1(x)$ and $F_2(x)$ are integrals of $f(x)$. Then, from Equation (2.7), we know that $F_1(x) = F_2(x) + c$. Therefore

$$\begin{aligned} F_1(b) - F_1(a) &= (F_2(b) + c) - (F_2(a) + c) \\ &= F_2(b) - F_2(a). \end{aligned}$$

So the difference between two integrals of $f(x)$ does not involve an arbitrary constant and, for any two given values a and b in its domain, is constant. If $F(x)$ is any integral of $f(x)$, we denote this constant by

$$[F(x)]_a^b = F(b) - F(a),$$

where $[F(x)]_a^b$ means that we evaluate $F(x)$ at $x = b$ and subtract from this the value of $F(x)$ at $x = a$. We call this quantity the *definite integral of* $f(x)$ *from* $x = a$ *to* $x = b$. This definite integral is written $\int_a^b f(x)\,dx$, so we have

We may also refer to $\int_a^b f(x)\,dx$ as the *definite integral of* $f(x)$ *over the interval* $[a, b]$.

$$\int_a^b f(x)dx = [F(x)]_a^b. \tag{2.8}$$

The numbers a and b are called, respectively, the *lower limit of integration* and the *upper limit of integration*.

Definition

The **definite integral** of $f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx$, where a and b are the **lower limit of integration** and the **upper limit of integration**, respectively. If $F(x)$ is any **integral** of the function $f(x)$,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a). \tag{2.8}$$

The procedure for evaluating the definite integral $\int_a^b f(x)\,dx$ is as follows.

- Find *any* integral $F(x)$ of the function $f(x)$.
- Subtract the value of $F(x)$ at the lower limit of integration $x = a$ from its value at the upper limit $x = b$.

The indefinite integral will do instead of 'any integral', but because of the second step the '$+c$' is not needed here.

Exercise 2.4

Calculate to two decimal places the values of the following definite integrals.

(a) $\displaystyle\int_0^1 \sin(\pi x)\,dx.$ (b) $\displaystyle\int_{-1}^3 e^{4x} + x^3\,dx.$ (c) $\displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 t\,dt.$

(d) $\displaystyle\int_{-1}^1 \frac{1}{1+u^2}\,du.$ (e) $\displaystyle\int_0^1 e^x\,dx.$

Since definite integration involves evaluation of the function at the two limits, the choice of label (if any) for the variable of integration has no bearing on the outcome. For example,

Such a variable is said to be a *dummy variable*.

$$\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx \quad \text{and} \quad \int_{-\pi/4}^{\pi/4} \sec^2 u\,du$$

are the same.

Certain properties of the definite integral follow from the definition in Equation (2.8).

Exercise 2.5

Use the definition of the definite integral to establish the following results, where a, b and c are any values in the domain of the function $f(x)$.

(a) $\displaystyle\int_a^a f(x)\,dx = 0.$ (b) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

(c) $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$

The Constant Multiple Rule and Sum Rule apply for definite integrals as well as for indefinite integrals, so

$$\int_a^b (kf(x)) = k\int_a^b f(x)\,dx,$$

where k is any constant, and

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Definite integrals and areas

The definite integral has an important interpretation in terms of area, which arises from concentrating on the graph of $f(x)$ rather than on that of $F(x)$. The following argument establishes this link.

Suppose that $f(x) \geq 0$ for all x in the interval $[a, b]$, so that its graph never dips below the x-axis on this interval. Let $A(x)$ be defined as the area beneath the graph of $f(x)$ from a to x, where $x < b$. That is, $A(x)$ is the area bounded by the x-axis, the curve $y = f(x)$ and the vertical lines through a and x on the x-axis, as illustrated in Figure 2.1.

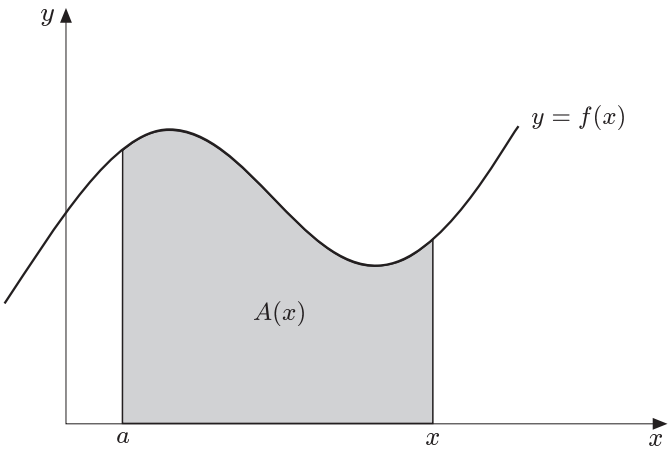


Figure 2.1 Area $A(x)$ is shaded

First, note that $A(a) = 0$, since, if $x = a$, the 'area' $A(x)$ has zero width.

Now consider the area $A(x + h)$, for which the right-hand bounding vertical line is moved a distance h to the right. Figure 2.2 shows the area between the vertical lines at x and at $x + h$, which is equal to

$$A(x + h) - A(x).$$

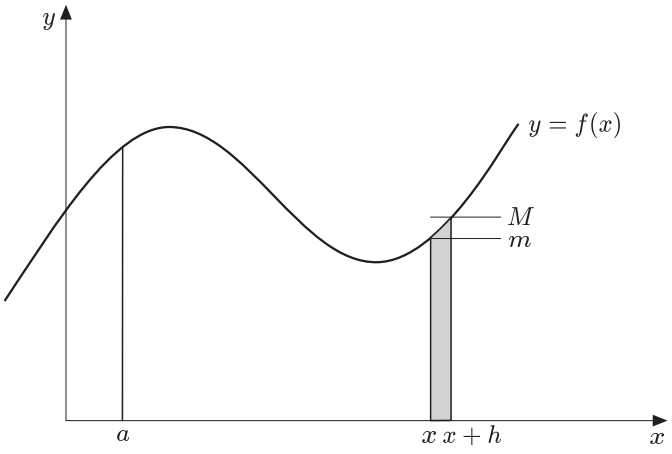


Figure 2.2 Area $A(x + h) - A(x)$ is shaded

Now, over the interval $[x, x + h]$, the function $f(x)$ lies between some minimum value m and some maximum value M. The rectangle with height m and width h has area no greater than that beneath the graph of $f(x)$ on this interval, while the rectangle with height M and width h has area no

less than that beneath the graph of $f(x)$. This can be expressed by the inequalities

$$mh \leq A(x+h) - A(x) \leq Mh$$

or, on dividing through by h ($h \neq 0$), as

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M.$$

We now take the limit of each of these expressions as $h \to 0$. By the definition of the derivative, we have

$$\lim_{h \to 0} \left(\frac{A(x+h) - A(x)}{h} \right) = A'(x).$$

As $h \to 0$, the strip between x and $x+h$ becomes vanishingly thin, and we would expect both m and M to tend to $f(x)$. This is true if $f(x)$ is continuous. We now have

$$f(x) \leq A'(x) \leq f(x)$$

and so

$$A'(x) = f(x).$$

Therefore

$$\int_a^x f(x)\,dx = A(x) - A(a) = A(x),$$

since $A(a) = 0$, so that the area between the graph of $y = f(x)$ and the x-axis, from a to x, is given by

$$A(x) = \int_a^x f(x)\,dx.$$

Thus, provided that $f(x) \geq 0$ for x in the interval $[a, b]$, the area beneath the graph of $f(x)$ between a and b is

$$A(b) = \int_a^b f(x)\,dx.$$

Finding such an area is therefore equivalent to evaluating the corresponding definite integral. For example, the shaded area in Figure 2.3(b) below is given by $\int_a^b g(x)dx$.

Exercise 2.6

(a) Find the area beneath the graph of $f(x) = e^{-\frac{1}{2}x}$ between $x = 0$ and $x = 4$, correct to four decimal places.

(b) Find the area beneath the graph of $y = \cos(3x)$ between $x = 0$ and $x = \frac{\pi}{6}$.

(c) Find the area bounded by the parabola $y = 1 - x^2$ and the x-axis.

(d) Write down (but do not evaluate) a definite integral to represent the area bounded by the circle $x^2 + y^2 = 1$ and by the positive x- and y-axes.

The last exercise illustrates how to find the area beneath the graph of a function $f(x)$ within an interval for which this graph lies above the x-axis. This leaves open the question of whether integration can also be used to describe an area of the type shown in Figure 2.3(a). The area shown is again bounded by the graph of $y = f(x)$ and by the x-axis, but here $f(x) \leq 0$ throughout the interval under consideration.

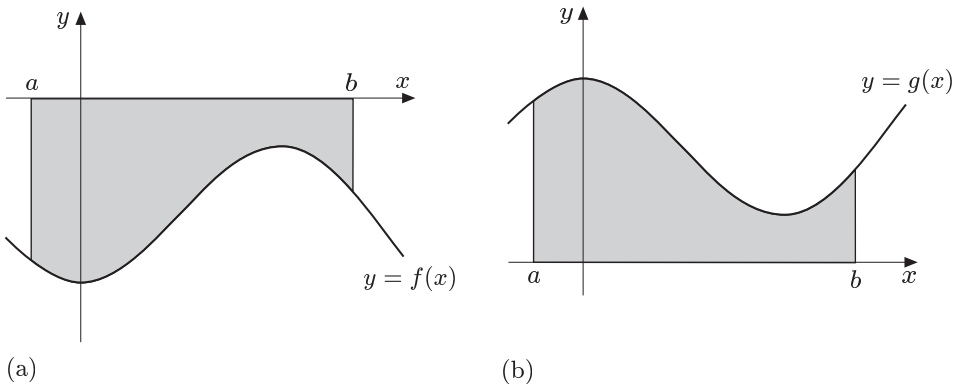


Figure 2.3 (a) Graph of a function $f(x)$ that is negative (b) Graph of $g(x) = -f(x)$ for $f(x)$ as in (a)

If $f(x) \leq 0$ for all x in the interval $[a, b]$, then the function $g(x)$, where $g(x) = -f(x)$, satisfies the condition that $g(x) \geq 0$ for all x in the interval $[a, b]$. The graph of $g(x)$ is shown in Figure 2.3(b). It is the reflection of the graph of $f(x)$ in the x-axis, from which it follows that the area marked beneath the graph of $g(x)$ is equal in size to the area with which we started in Figure 2.3(a). The area in Figure 2.3(b) is

$$\int_a^b g(x)\,dx = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

So we see that, for a function $f(x)$ with $f(x) \leq 0$ for all x in $[a, b]$, the area between the graph and the x-axis is equal to $-\int_a^b f(x)\,dx$. The definite integral $\int_a^b f(x)\,dx$ is negative and so, to obtain the area, which must be positive, we must change the sign.

We have now covered the cases where either $f(x)$ is non-negative over the whole interval of interest or non-positive over all of this interval. That leaves the general situation where $f(x)$ may take both positive and negative values in some interval, as illustrated in Figure 2.4.

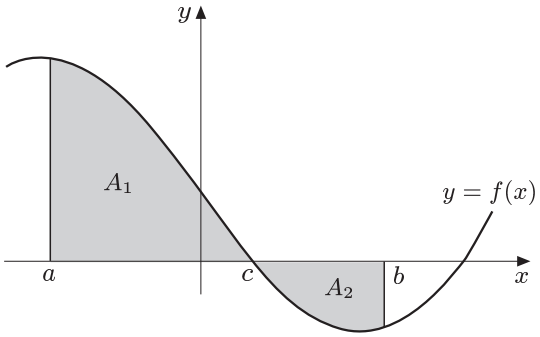


Figure 2.4 A function $f(x)$ that takes both positive and negative values in $[a, b]$

How can the integral

$$\int_a^b f(x)\,dx$$

be related in this case to the two areas A_1 and A_2 in Figure 2.4?

If the graph of $f(x)$ cuts the x-axis at $x = c$, as shown, then, by the result of Exercise 2.5(c), we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

But $f(x) \geq 0$ for x in the interval $[a, c]$, while $f(x) \leq 0$ for x in the interval $[c, b]$. From our earlier results, it follows that

$$A_1 = \int_a^c f(x)\,dx, \qquad A_2 = -\int_c^b f(x)\,dx.$$

We conclude that

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

This outcome generalizes to a function $f(x)$ which may cut the x-axis any number of times. In general:

- A_1 is the sum of the areas between $x = a$ and $x = b$ that are bounded below by the x-axis and above by the graph $y = f(x)$, in those intervals where $f(x) \geq 0$;
- A_2 is the sum of the areas between $x = a$ and $x = b$ that are bounded above by the x-axis and below by the graph $y = f(x)$, in those intervals where $f(x) \leq 0$.

Exercise 2.7

Find the total area enclosed by the graph of $y = \cos(3x)$ and the x-axis, between $x = 0$ and $x = \frac{\pi}{2}$.

2.2 Integration by parts

So far you have only two approaches available for finding indefinite integrals. The first is to use a table of standard integrals together with the Sum and Constant Multiple rules for integration. If the integrals of $f(x)$ and $g(x)$ are in the table, and a and b are constants, you can find $\int af(x) + bg(x)\,dx$. The other method open to you is 'guess and check'. Given a function $f(x)$, guess a function $F(x)$ that you think may be an integral, and then check whether $F'(x)$ is really equal to $f(x)$. Sometimes you can make this work by 'adjusting' $F(x)$ by multiplying by a constant, as illustrated in Example 2.1. In the next two subsections, you will meet two methods that provide a more systematic approach. It is useful to remember, though, that any expression you obtain for an indefinite integral can be checked by differentiation.

Most integrals that you will meet in MST209 can be evaluated by using the table of standard integrals.

In this subsection we introduce a technique of integration based on the Product Rule for differentiation.

Adapting the Product Rule

The Product Rule for differentiation states that if $f(x)$ and $g(x)$ are differentiable functions, then the derivative of their product is given by

$$\frac{d}{dx}(f(x) \times g(x)) = \frac{d}{dx}(f(x)) \times g(x) + f(x) \times \frac{d}{dx}(g(x)).$$

For example, if $f(x) = x$ and $g(x) = \sin x$, we may deduce from the Product Rule that

$$\begin{aligned}\frac{d}{dx}(x \sin x) &= \frac{dx}{dx} \times \sin x + x \times \frac{d}{dx}(\sin x)\\ &= \sin x + x \cos x.\end{aligned}$$

Noting that $-\cos x$ is an integral of $\sin x$, this may be expressed as

$$\frac{d}{dx}(x\sin x) = \frac{d}{dx}(-\cos x) + x\cos x$$

or, rearranging this equation, as

$$x\cos x = \frac{d}{dx}(x\sin x + \cos x).$$

This equation states that $x\sin x + \cos x$ is an integral of $x\cos x$, which can be written using the indefinite integral notation as

$$\int x\cos x\,dx = x\sin x + \cos x + c,$$

where c is an arbitrary constant.

So, knowing the derivative of the product $x\sin x$ has enabled us to find the indefinite integral of a different product $x\cos x$. In deriving this indefinite integral, we relied on recognizing that $-\cos x$ is an integral of $\sin x$. For an effective general procedure, we need to systematize this step. To see how such an approach might work more generally, we retrace the steps of the argument above which led to finding $\int x\cos x\,dx$, but this time retain full generality, with $f(x)$ in place of x and $g(x)$ in place of $\sin x$. The starting point is the Product Rule,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Suppose that $H(x)$ is an integral of $f'(x)g(x)$, so that $H'(x) = f'(x)g(x)$. Then the Product Rule can be expressed as

$$(f(x)g(x))' = H'(x) + f(x)g'(x)$$

or, rearranging and then applying the Sum Rule for differentiation and the Constant Multiple Rule for differentiation, as

$$\begin{aligned} f(x)g'(x) &= (f(x)g(x))' - H'(x) \\ &= (f(x)g(x) - H(x))'. \end{aligned}$$

This equation states that $f(x)g(x) - H(x)$ is an integral of $f(x)g'(x)$ and so can be written using the indefinite integral notation as

$$\int f(x)g'(x)\,dx = f(x)g(x) - H(x) + c,$$

where c is an arbitrary constant. But $H(x)$ is an integral of $f'(x)g(x)$ and so this may also be interpreted as

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{2.9}$$

The application of this formula provides a method of integration known as **integration by parts**. Equation (2.9) permits the integration of certain products to be carried out in stages. It does not work for all products, by any means. The formula converts the problem of integrating $f(x)g'(x)$ into the problem of integrating $f'(x)g(x)$. If, for particular functions $f(x)$ and $g(x)$, it is possible to find $\int f'(x)g(x)\,dx$, then Equation (2.9) enables us to find $\int f(x)g'(x)\,dx$. For this approach to be worthwhile, the product $f'(x)g(x)$ must be simpler to integrate than $f(x)g'(x)$.

Integration by parts for indefinite integrals

This method attempts to integrate $f(x)g'(x)$ using the formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx. \tag{2.9}$$

For the method to be worthwhile, the product $f'(x)g(x)$ must be simpler to integrate than the original product $f(x)g'(x)$.

Example 2.2

Use integration by parts to find the indefinite integral

$$\int x \sin x\,dx.$$

Solution

In order to apply Equation (2.9), we need to choose functions $f(x)$ and $g'(x)$ so that

$$\int x \sin x\,dx = \int f(x)g'(x)\,dx.$$

This is achieved if we take

$$f(x) = x, \qquad g'(x) = \sin x.$$

This choice of $f(x)$ and $g'(x)$ implies that

$$f'(x) = 1, \qquad g(x) = -\cos x.$$

We could have taken $g(x) = -\cos x + 1$ or, indeed, $g(x) = -\cos x + c$ for any value of the constant c. However, we are at liberty to choose the most convenient form for $g(x)$. The final expression for $\int x \sin x\,dx$ must contain an arbitrary constant.

Using Equation (2.9), we obtain

$$\begin{aligned}\int x \sin x\,dx &= x(-\cos x) - \int 1 \times (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + c. \quad \blacksquare\end{aligned}$$

Exercise 2.8

Use integration by parts to find the following indefinite integrals.

(a) $\int x \cos x\,dx.$ (b) $\int xe^x\,dx.$

As observed earlier, the formula used for integration by parts converts the problem of integrating the product $f(x)g'(x)$ into the problem of integrating $f'(x)g(x)$. To apply the technique, therefore, we must first of all recognize that we are integrating a product. Then this product must be expressed as $f(x)g'(x)$ by choosing which part of the product to call $f(x)$ and which $g'(x)$. In making this choice, the following two guiding principles apply.

- You must choose as $g'(x)$ a function for which you can find $g(x)$; that is, you must be able to integrate your chosen function $g'(x)$.
- The resulting integral $\int f'(x)g(x)\,dx$ should be simpler than the integral with which you started.

Exercise 2.9

(a) Use integration by parts to find

$$\int x \ln x \, dx.$$

You should find that the first of the two guiding principles is useful.

(b) Use your solution to (a) to evaluate the definite integral

$$\int_1^2 x \ln x \, dx,$$

giving your answer correct to four decimal places.

Exercise 2.10

Use integration by parts to find the following indefinite integrals.

(a) $\int x \cos(2x) \, dx.$ (b) $\int x \sin(3x) \, dx.$

Exercise 2.11

(a) Use integration by parts to find

$$\int xe^{2x} \, dx.$$

(b) Explain why the graph of $y = xe^{2x}$ is above the x-axis for values of x in the interval $[1, 2]$ and write down a definite integral that gives the area under this graph between $x = 1$ and $x = 2$.

(c) Use your result from (a) to evaluate the area described in (b), giving your answer correct to four decimal places.

When finding a definite integral using integration by parts, we can first find the indefinite integral, then apply the limits of integration. It is usually more convenient to apply the limits 'as you go along' and this can be done using the following result.

Integration by parts for definite integrals

$$\int_a^b f(x)g'(x) \, dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) \, dx. \qquad (2.10)$$

Example 2.3

Use Equation (2.10) to evaluate the definite integral

$$\int_1^2 xe^{2x} \, dx.$$

You evaluated the definite integral in Exercise 2.11(c).

Solution

We proceed as follows. Let $f(x) = x$ and $g'(x) = e^{2x}$, so that $f'(x) = 1$ and $g(x) = \frac{1}{2}e^{2x}$.

Substitution into Equation (2.10) gives

$$\begin{aligned}
\int_1^2 xe^{2x} \, dx &= \left[\tfrac{1}{2}xe^{2x}\right]_1^2 - \tfrac{1}{2}\int_1^2 e^{2x} \, dx \\
&= \left(e^4 - \tfrac{1}{2}e^2\right) - \tfrac{1}{2}\left[\tfrac{1}{2}e^{2x}\right]_1^2 \\
&= e^4 - \tfrac{1}{2}e^2 - \tfrac{1}{4}e^4 + \tfrac{1}{4}e^2 \\
&= \tfrac{3}{4}e^4 - \tfrac{1}{4}e^2 \\
&= \tfrac{1}{4}(3e^4 - e^2) \\
&= 39.1013 \quad \text{(to four decimal places).} \quad \blacksquare
\end{aligned}$$

In Example 2.3 we gave the answer correct to four decimal places. To quote a value correct to four decimal places it is necessary to work to at least six decimal places to avoid rounding errors decreasing the accuracy. The value given in the example was obtained using the calculator's full accuracy.

Exercise 2.12

Use the integration by parts formula for definite integrals to find

$$\int_0^1 xe^{3x}\,dx,$$

giving your answer correct to four decimal places.

Exercise 2.13

(a) Explain why the graph of $y = x\sin(3x)$ is above the x-axis for values of x in the interval $[\frac{\pi}{12}, \frac{\pi}{6}]$, and write down a definite integral that gives the area under this graph between $x = \frac{\pi}{12}$ and $x = \frac{\pi}{6}$.

(b) Use the integration by parts formula for definite integrals to evaluate the area described in (a), giving your answer correct to four decimal places.

2.3 Integration by substitution

As stated before, most of the integrals in MST209 can be obtained by utilizing the table of standard integrals.

In Subsection 2.2 we derived the rule for integration by parts through adapting the Product Rule for differentiation. In this subsection we develop another technique to extend the range of integrals that can be done by hand. This time we shall use the Composite Rule for differentiation, which we can write in the form

$$k'(x) = h'(g(x)) \times g'(x),$$

where $k(x) = h(g(x))$.

Reversing this equation and integrating both sides, we obtain

$$\begin{aligned}\int h'(g(x)) \times g'(x)\,dx &= \int k'(x)\,dx \\ &= k(x) + c \\ &= h(g(x)) + c.\end{aligned}$$

Suppose that $h(u)$ is an integral of some function $f(u)$, so that we have $h(u) = \int f(u)\,du$ and $h'(u) = f(u)$. Then, putting $u = g(x)$, we can recast the last equation in a more useful form as

$$\begin{aligned}\int f(g(x)) \times g'(x)\,dx &= h(g(x)) + c \\ &= h(u) + c \\ &= \int f(u)\,du. \qquad (2.11)\end{aligned}$$

To use this result, the function to be integrated needs to be of a suitable form. It needs to be the product of two terms, one a composite function and the other the derivative of the 'inner' function $g(x)$. Symbolically, this general form is

$$f(g(x)) \times g'(x).$$

Then Equation (2.11) states that integration of this product can be reduced to integration of the 'outer' function $f(x)$.

In using Equation (2.11), we must remember that u stands for the inner function $g(x)$ and so we need to replace u by $g(x)$ in the expression obtained for $\int f(u)\,du$. For example, to integrate $3x^2\cos(x^3)$ with respect to x using Equation (2.11), we choose $u = x^3$, since $du/dx = 3x^2$, and so we have

$$\begin{aligned}\int 3x^2\cos(x^3)\,dx &= \int \cos(x^3)(3x^2)\,dx \\ &= \int \cos u\,du \\ &= \sin u + c \\ &= \sin(x^3) + c.\end{aligned}$$

The *outer function* is $f(x) = \cos x$, the *inner function* is $g(x) = x^3$ and the *composite function* is $f(g(x)) = \cos(x^3)$.

In the calculation of $\int 3x^2\cos(x^3)\,dx$ above, we set $u = x^3$ and, in the second step, we replaced $3x^2\,dx$ by du. Thus, in effect, we replaced $(du/dx)dx$ by du. It looks as if we are cancelling the dxs and this provides a useful way to remember the method (although this is not mathematically correct since du/dx is not a quotient).

The reason for the name of this method of integration now becomes apparent. We *substitute* a single variable (u here) for the function $g(x)$. The approach to integration by substitution in particular examples may then be streamlined by omitting any explicit mention of the functions $f(x)$ and $g(x)$. We work the next example using this streamlined approach. We proceed in six steps. As described below, the method assumes that we want to find an integral given in terms of x and that we use u as the new variable. Other variable names could, of course, be used instead of these.

Procedure 2.1 Integration by substitution for indefinite integrals

To evaluate an indefinite integral of the form

$$k\int f(g(x)) \times g'(x)\,dx,$$

where k is a constant, follow the steps given below.

(a) Choose $u = g(x)$.

(b) Find du/dx.

(c) Replace $(du/dx)dx$ by du and write the rest of the integrand in terms of u.

(d) Integrate $k\int f(u)\,du$.

(e) Substitute back for $u = g(x)$.

Example 2.4

Find the indefinite integral

$$\int x^2(x^3+1)^8\,dx.$$

Solution

(a) Choose u. Here we let

$$u = x^3 + 1.$$

This choice of u is guided by x^3+1 being the inner function in the composite function $(x^3+1)^8$.

(b) Find du/dx. Here we have

$$\frac{du}{dx} = 3x^2.$$

(c) Substitute for x in terms of u:

$$\begin{aligned}\int x^2(x^3+1)^8\,dx &= \int \tfrac{1}{3}(x^3+1)^8(3x^2)\,dx \\ &= \int \tfrac{1}{3}u^8\,du.\end{aligned}$$

(d) Integrate $k\int f(u)\,du$:

$$\tfrac{1}{3}\int u^8\,du = \tfrac{1}{3}\times\tfrac{1}{9}u^9 + c,$$

where c is an arbitrary constant.

(e) Substitute back:

$$\tfrac{1}{3}\times\tfrac{1}{9}u^9 + c = \tfrac{1}{27}(x^3+1)^9 + c.$$

We conclude that

$$\int x^2(x^3+1)^8\,dx = \tfrac{1}{27}(x^3+1)^9 + c. \quad \blacksquare$$

For this method of integration by substitution to succeed, the integral obtained at Step (d) must be simpler than the original and the choice of u in terms of x is made with this in mind. Skill at picking a suitable substitution (and there may be more than one that is suitable) comes with practice and experience.

The integrals in Exercise 2.14 give you practice in using the method. You are given a suitable substitution for u in terms of x in each case, where each u is the inner part of a composite function.

Exercise 2.14

Use integration by substitution to find the following integrals.

(a) $\int (5+2x^2)^{16}x\,dx$, taking $u = 5+2x^2$.

(b) $\int x^2\sec^2(x^3)\,dx$, taking $u = x^3$.

(c) $\int (\sin x)\,e^{\cos x}\,dx$, taking $u = \cos x$.

(d) $\displaystyle\int \frac{1}{1+9x^2}\,dx$, taking $u = 3x$.

Where a composite function is of the form $f(g(x))$, with $f(u) = 1/u$, the composite may not be immediately recognized as such.

Exercise 2.15

Find $\displaystyle\int \frac{x}{1+x^2}\,dx$, taking $u = 1 + x^2$.

It is useful to know that any integral of the form

$$k \int \frac{g'(x)}{g(x)}\,dx$$

We assume here that $g(x) > 0$.

can be evaluated by taking $u = g(x)$, leading to the solution $\ln(g(x)) + c$.

Now you are invited to try applying integration by substitution in cases where an appropriate substitution is not provided.

Exercise 2.16

Use integration by substitution to find the following integrals.

(a) $\displaystyle\int (6 + x^3)^9 x^2\,dx$. (b) $\displaystyle\int e^{2x} \sin(e^{2x})\,dx$. (c) $\displaystyle\int \sin^4 x \cos x\,dx$.

(d) $\displaystyle\int x\sqrt{1+x^2}\,dx$. (e) $\displaystyle\int \frac{x^2}{1+x^3}\,dx \quad (x > -1)$.

So far you have applied integration by substitution to indefinite integrals. If the value of a definite integral is required, this may be obtained after finding the indefinite integral (or any integral) in the usual way, by subtracting its value at the lower limit of integration from its value at the upper limit of integration. However, there is a version of the integration by substitution formula that applies directly to definite integrals and may shorten the calculation.

When the substitution is used to replace a definite integral in terms of x by a definite integral in terms of u, the limits of integration must be appropriately substituted. While this is a separate step, it means that the remaining integral can be evaluated directly without the need to substitute back for u in terms of x.

The formula for definite integrals arises from the fact that, if $u = g(x)$,

$$[F(g(x))]_a^b = F(g(b)) - F(g(a)) = [F(u)]_{g(a)}^{g(b)}.$$

If $F'(g(x)) = f(g(x))$, the Composite Rule for differentiation tells us that $F(g(x))$ is an integral of $f(g(x)) \times g'(x)$, so

$$\begin{aligned}\int_a^b f(g(x)) \times g'(x)\,dx &= [F(g(x))]_a^b \\ &= [F(u)]_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du. \end{aligned} \tag{2.12}$$

Procedure 2.2 Integration by substitution for definite integrals

To evaluate a definite integral of the form

$$k\int_a^b f(g(x)) \times g'(x)\,dx,$$

where k is a constant, follow the steps given below.

(a) Choose $u = g(x)$.

(b) Find du/dx.

(c) Replace $(du/dx)dx$ by du, write the rest of the integrand in terms of u, and replace the limits of integration a and b by $g(a)$ and $g(b)$, respectively.

(d) Integrate and evaluate $k\int_{g(a)}^{g(b)} f(u)\,du$.

Example 2.5

Evaluate the definite integral

$$\int_0^1 e^{2x}\sqrt{1+e^{2x}}\,dx.$$

Solution

(a) For the substitution, we choose

$$u = 1 + e^{2x}.$$

(b) The derivative is

$$\frac{du}{dx} = 2e^{2x}.$$

(c) The integral can be written to include du/dx explicitly as

$$\int_0^1 e^{2x}\sqrt{1+e^{2x}}\,dx = \int_0^1 \tfrac{1}{2}\sqrt{1+e^{2x}}(2e^{2x})\,dx.$$

We turn next to the limits of integration:

when $x = 0,\ u = 1 + e^{2\times 0} = 2;$

when $x = 1,\ u = 1 + e^{2\times 1} = 1 + e^2.$

We use Equation (2.12) with $a = 0$, $b = 1$ and $g(x) = 1 + e^{2x}$. Hence $g(a) = 2$ and $g(b) = 1 + e^2$.

So the limits of integration for u are 2 and $1 + e^2$.

Now we have

$$\begin{aligned}\int_0^1 \tfrac{1}{2}\sqrt{1+e^{2x}}(2e^{2x})\,dx &= \int_0^1 \tfrac{1}{2}\sqrt{u}\,\frac{du}{dx}\,dx \\ &= \int_2^{1+e^2} \tfrac{1}{2}u^{\frac{1}{2}}\,du.\end{aligned}$$

(d) Therefore, we have

$$\begin{aligned}\int_0^1 e^{2x}\sqrt{1+e^{2x}}\,dx &= \tfrac{1}{2}\int_2^{1+e^2} u^{\frac{1}{2}}\,du \\ &= \tfrac{1}{2}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_2^{1+e^2} \\ &= \tfrac{1}{3}((1+e^2)^{\frac{3}{2}} - 2^{\frac{3}{2}}) \\ &= 7.1565 \quad \text{(to four decimal places).}\end{aligned}$$ ■

Exercise 2.17

Use integration by substitution to evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{x}{1+2x^2}\,dx$, taking $u = 1 + 2x^2$.

(b) $\displaystyle\int_0^{\pi/2} \frac{\sin x \cos x}{1+\cos^2 x}\,dx$, taking $u = 1 + \cos^2 x$.

(c) $\displaystyle\int_{-1}^0 xe^{-3x^2-2}\,dx$, taking $u = -3x^2 - 2$.

(d) $\displaystyle\int_0^{\pi/4} \sec^3 x\,\tan x\,dx$, taking $u = \sec x$.

End-of-section Exercises

Exercise 2.18

Find the following indefinite integrals.

(a) $\displaystyle\int x^{18} + x\,dx$. (b) $\displaystyle\int -\frac{6}{x^4}\,dx$. (c) $\displaystyle\int \sin\left(\tfrac{1}{3}x\right)\,dx$.

(d) $\displaystyle\int e^{-2x}\,dx$. (e) $\displaystyle\int \sec^2 x\,dx$. (f) $\displaystyle\int \frac{(3x^2+1)(x-1)}{\sqrt{x}}\,dx$.

Where integrals involve expressions that are not defined for all x in $\mathbb{R}$, we assume that we are working in a suitable interval.

Exercise 2.19

Calculate the following definite integrals

(a) $\displaystyle\int_0^1 7x^8\,dx$. (b) $\displaystyle\int_0^{\pi/4} \cos(2x)\,dx$.

(c) $\displaystyle\int_{-1}^1 e^{7x}\,dx$. (d) $\displaystyle\int_{\pi/4}^{\pi/3} \operatorname{cosec}^2 x\,dx$.

(e) $\displaystyle\int_0^{\pi/4} \sin^2 x\,dx$, given that $\cos(2x) = 1 - 2\sin^2 x$.

Exercise 2.20

(a) Find the area between the curve $y = x^2 - 4x$ and the x-axis from $x = -1$ to $x = 0$.

(b) Find the area between the curve $y = x^2 - 4x$ and the x-axis from $x = 0$ to $x = 2$.

(c) Explain why the sum of the areas found in (a) and (b) is not equal to $\int_{-1}^2 x^2 - 4x\,dx$.

Exercise 2.21

Use integration by parts to find the following integrals.

(a) $\int xe^{-x}\,dx$. (b) $\int \sqrt{x}\ln x\,dx \quad (x>0)$.

(c) $\int x\cos(4x)\,dx$. (d) $\int_0^1 xe^{-x}\,dx$. (Use your answer to (a).)

Exercise 2.22

Use the given substitution to find the following integrals.

(a) $\int x(1+x^2)^3\,dx$, taking $u=1+x^2$.

(b) $\int \sin x\cos^3 x\,dx$, taking $u=\cos x$.

(c) $\int \dfrac{x-2}{x^2-4x+3}\,dx$, taking $u=x^2-4x+3 \quad (x>3)$.

(d) $\int \dfrac{x^5}{1+x^3}\,dx$, taking $u=1+x^3 \quad (x>-1)$.

Exercise 2.23

Use integration by substitution to find the following integrals.

(a) $\int x(1+x^2)^5\,dx$. (b) $\int \cos x\sin^3 x\,dx$.

(c) $\int (24x+9)(4x^2+3x+5)^7\,dx$. (d) $\int \dfrac{6x+4}{\sqrt{3x^2+4x-1}}\,dx \quad (x>1)$.

Exercise 2.24

Use the given substitution to find the following integrals.

(a) $\int \dfrac{x}{\sqrt{x-2}}\,dx$, taking $u=\sqrt{x-2} \quad (x>2)$.

(b) $\int_4^9 \dfrac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx$, taking $u=\sqrt{x}-1$.

(c) $\int \dfrac{1}{\sqrt{9-x^2}}\,dx$, taking $u=\arcsin\left(\frac{1}{3}x\right) \quad (-3<x<3)$.

(d) $\int \dfrac{1}{9+x^2}\,dx$, taking $u=\arctan\left(\frac{1}{3}x\right)$.

Exercise 2.25

Use integration by substitution to find the following integrals.

The substitutions are less easy to see here.

(a) $\int \dfrac{\sec^2 x}{\tan x}\,dx \quad (0<x<\frac{\pi}{2})$. (b) $\int \dfrac{x^3}{(1+x^2)^{\frac{1}{2}}}\,dx$.

(c) $\int \dfrac{1}{4x^2+9}\,dx$. (d) $\int \dfrac{e^x}{\sqrt{1-e^{2x}}}\,dx \quad (x<0)$.

Exercise 2.26

Find the following integrals, using any appropriate method.

(a) $\int x^3\cos(5x^4)\,dx$. (b) $\int x\cos(5x)\,dx$. (c) $\int x+\cos(5x)\,dx$.

(d) $\int \dfrac{1}{1+2x^2}\,dx$. (e) $\int \dfrac{x^2}{(7-x^3)^7}\,dx \quad \left(x<\sqrt[3]{7}\right)$. (f) $\int xe^{-\frac{1}{3}x}\,dx$.

3 Complex numbers

Complex numbers are more than just a mathematical curiosity. To engineers and physicists they are a tool as versatile and useful as one of those multi-bladed pocket knives. Some of us use them daily — especially in the analysis of electric circuits and of waves, whether surface waves on ponds, sound waves in air, or electromagnetic waves within, around and between planets and stars. Though you will not get all the way there within MST209, this could be the first step along an exciting and enlightening road.

Subsection 3.1 gives an historical overview of how complex numbers first arose. Then, in Subsection 3.2, we look at the arithmetic of complex numbers and define the complex conjugate. One of the interesting features is that all the arithmetic operations have a geometric interpretation. This is developed further in Subsection 3.3 where we meet the polar form, which is particularly convenient for multiplication. In Subsection 3.4 we look at the exponential function applied to complex numbers and derive Euler's famous formula.

3.1 Origins of complex numbers

Italian mathematicians of the sixteenth century, exploring techniques for solving algebraic problems, came up with numbers like 'the square root of minus fifteen'. This is where our present story really starts.

Many extensions to the idea of number can be associated with a wish to be able to solve certain types of equation. To solve the equations

$$x + 5 = 0 \quad \text{and} \quad x^2 = 2,$$

for example, we need, for the first, negative numbers and, for the second, irrational numbers. In this section we look at the solution of equations such as

$$x^2 + 1 = 0.$$

To do this we need to introduce yet more 'numbers'. Mathematicians found these new numbers easy to use, but were concerned for a long time about their 'reality'. These numbers were finally put on a firm footing in the nineteenth century — but let us start the story at its beginning.

In trying to find a rectangle with perimeter 20 and area 40, Gerolamo Cardano (1501–1576) introduced two variables — one for the width and one for the height. If we call these variables x and y, the problem requires trying to solve the simultaneous equations

$$\left.\begin{aligned} x + y &= 10, \\ xy &= 40. \end{aligned}\right\} \tag{3.1}$$

Eliminating the variable y, the problem reduces to solving the quadratic equation

$$x^2 - 10x + 40 = 0.$$

Now any quadratic equation

$$ax^2 + bx + c = 0$$

can be solved using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In Cardano's example, with $a = 1$, $b = -10$ and $c = 40$, the formula gives one solution as $x = 5 + \sqrt{-15}$ and it follows that $y = 10 - x = 5 - \sqrt{-15}$.

Even though these solutions involve $\sqrt{-15}$, the square root of a negative number, Cardano found that he could treat them like any real number.

To show that he had solved Equations (3.1), he argued as follows. Substituting in the first equation gives

$$x + y = 5 + \sqrt{-15} + 5 - \sqrt{-15} = 10,$$

substituting in the second equation gives

$$xy = (5 + \sqrt{-15})(5 - \sqrt{-15}) = 5^2 - (\sqrt{-15})^2 = 25 - (-15) = 40,$$

and both are as required. In doing this, all he had to assume was that squaring undoes the square root operation, regardless of the number involved; that is, that $(\sqrt{-15})^2 = -15$.

Although this in some sense 'solved' his original problem, not even Cardano felt comfortable about his use of $\sqrt{-15}$. (He invited treatment of expressions involving it as if they were ordinary numbers 'leaving aside the mental tortures involved'.)

Later mathematicians continued to treat the occurrence of such 'numbers' with suspicion. In 1637 René Descartes (1596–1650) still felt that they were 'imaginary numbers, for clearly they are creatures of the imagination'.

Exercise 3.1

Evaluate the product $(3 + \sqrt{-2})(3 - \sqrt{-2})$.

3.2 Using complex numbers

In this section we shall develop an arithmetic of numbers such as $5 + \sqrt{-15}$, involving the square roots of negative numbers.

Notation

For any positive real number r, we can write

$$\sqrt{-r} = \sqrt{r(-1)} = \sqrt{r}\sqrt{-1},$$

where $\sqrt{r}$ is taken to mean the positive square root. Since $r > 0$, $\sqrt{r}$ is a real number, so we can express the square roots of all negative numbers as a real multiple of the square root of -1. Leonhard Euler (1707–1783) introduced the symbol i for $\sqrt{-1}$. Using this we have, for example, $\sqrt{-16} = \sqrt{16}i = 4i$.

With this notation we can, for example, write $5 + \sqrt{-15}$ as $5 + \sqrt{15}i$. So using i, we can represent all the new numbers with which we shall be concerned.

Definitions

(a) A **complex number** is of the form $a + bi$, where a and b may be any real numbers and where $i^2 = -1$.

The set of all complex numbers is represented by $\mathbb{C}$.

(b) For a complex number $z = a + bi$, the **real part** of z is a and the **imaginary part** of z is b. The real part a is designated by $\mathrm{Re}(z)$ and the imaginary part b is designated by $\mathrm{Im}(z)$.

The imaginary part of a complex number is a real number. So, for example, the imaginary part of the complex number $3 - 5i$ is -5, not $-5i$.

Examples of complex numbers are $3+4i$, $12-5i$ and $-1-2i$. However, there is no restriction on the real numbers that can be used as the real and imaginary parts. They can be rationals like $\frac{3}{2}$, roots like $\sqrt{2}$, numbers like π, or any combination of such real numbers. In particular, $0+2i$, $-3+0i$ and $0+0i$ are all complex numbers. For simplicity, we write these numbers as $2i$, -3 and 0, respectively, omitting 0s where possible. We usually write numbers like $3+1i$ and $-1-1i$ as $3+i$ and $-1-i$, respectively, suppressing the redundant symbol 1.

This notation demonstrates that the real numbers are included in the set of complex numbers. For example, the real number 5 is identified with the complex number $5+0i$.

Exercise 3.2

(a) What are the real and imaginary parts of $2i$ and -3?

(b) Solve the quadratic equation $z^2-6z+25=0$, giving the answers as the complex numbers z_1 and z_2. Write down the real and imaginary parts of these numbers.

We sometimes write the imaginary part of a complex number after the i. For example, we may write $1+i\sqrt{2}$ in preference to $1+\sqrt{2}i$. This is often done to avoid any possible confusion between $\sqrt{2}i$ and $\sqrt{2i}$. Similarly we may write $i\sin\theta$ rather than $\sin\theta\, i$.

Arithmetic of complex numbers

So far we have performed arithmetic on complex numbers as if they were real numbers. This turns out to be justified, but it would be incautious simply to assume that it is. To be confident that this is correct, we should be systematic.

First we must define just what we mean by addition and multiplication of complex numbers. Then, using our definitions, we should verify that these arithmetic operations on complex numbers do share various properties with the corresponding operations on real numbers, which form the basis of ordinary arithmetic.

Examples of the sort of properties that we are referring to are

$$(a+b)+c=a+(b+c),$$

which states that, in adding three real numbers, the order in which the two additions are performed does not matter, and

$$a\times(b+c)=a\times b+a\times c,$$

which we use when expanding brackets.

Complex numbers are built from their real and imaginary parts, so two complex numbers are equal if their real parts are equal and their imaginary parts are equal.

Equality of complex numbers
If z and w are two complex numbers, then $z=w$ if and only if $\mathrm{Re}(z)=\mathrm{Re}(w)$ and $\mathrm{Im}(z)=\mathrm{Im}(w)$.

We next look at addition and multiplication of complex numbers.

Consider the addition of the two complex numbers $3+4i$ and $12+5i$:

$$\begin{aligned}(3+4i)+(12+5i)&=(3+12)+(4i+5i)\\&=15+(4+5)i\\&=15+9i.\end{aligned}$$

Here we have added the real parts and added the imaginary parts.

Addition of complex numbers
If $z = a + bi$ and $w = c + di$, then

$$z + w = (a + c) + (b + d)i.$$

It is not necessary to remember this formal definition. Addition can be performed from first principles.

Exercise 3.3

Add the two complex numbers $-3 + 4i$ and $12 - 5i$.

Multiplying two complex numbers is only slightly harder. We expand the expression for the product in the usual way and simplify using $i^2 = -1$. For example,

$$\begin{aligned}(3 + 4i) \times (12 + 5i) &= 3 \times 12 + 3 \times 5i + 12 \times 4i + 4 \times 5i^2 \\ &= 36 + 15i + 48i - 20 \\ &= 16 + 63i.\end{aligned}$$

This illustrates the method of multiplying complex numbers, which is defined below.

Multiplication of complex numbers
If $z = a + bi$ and $w = c + di$, then

$$z \times w = (ac - bd) + (ad + bc)i.$$

Again there is no need to remember this formula. Like addition, multiplication of complex numbers can be performed from first principles. We need to remember only that $i^2 = -1$.

Exercise 3.4

(a) Find the product $(-3 + 2i) \times (6 - 5i)$.

(b) Expand the general product $(a + bi) \times (c + di)$ and verify that the expression in the definition of multiplication of complex numbers given in the box above is obtained.

Both addition and multiplication are **commutative**; that is, if z_1 and z_2 are any two complex numbers,

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1.$$

The order of operation does not matter and you can easily check that this is true. You can also check that both addition and multiplication are **associative**; that is, if z_1, z_2 and z_3 are three complex numbers

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2)z_3 = z_1(z_2 z_3).$$

Similarly you can check that multiplication is **distributive** over addition; that is,

$$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3.$$

We now turn to subtraction and division of complex numbers. It would seem natural that, say,

$$(4 + 6i) - (5 + 2i) = (4 - 5) + (6 - 2)i = -1 + 4i.$$

Indeed, as we would expect and desire, $z - w$ is obtained by simply subtracting each of the real and imaginary parts of w from those of z.

Subtraction of complex numbers
If $z = a + bi$ and $w = c + di$, then

$$z - w = (a - c) + (b - d)i.$$

Exercise 3.5

If $z = -3 + 4i$ and $w = 12 - 5i$, find $z - w$.

With the above definition of subtraction, we can perform the usual arithmetic manipulations. For instance, if z, w and u are three complex numbers and $z = u + w$, it follows that $z - w = u$.

We shall now move on to look at division. The value of

$$\frac{4 + 6i}{5 + 2i}$$

is less obvious. To deal with the division z/w of the complex numbers z and w, we proceed in two steps. We first define the reciprocal $1/w$ of a complex number w and then define z/w as $z \times 1/w$.

For each real number $a \neq 0$, there is a number a^{-1} that satisfies

$$a \times a^{-1} = 1$$

and is called the **reciprocal** of a.

The reciprocal of a, $a \neq 0$, is $1/a$.

So, for a complex number z with $z \neq 0$, we would want the reciprocal z^{-1} to be the complex number for which

$$z \times z^{-1} = 1. \tag{3.2}$$

Then we can define w/z to be $w \times z^{-1}$.

Consider $z = 3 + 4i$. Suppose z has reciprocal $z^{-1} = x + yi$. From Equation (3.2), we know that

$$1 = z \times z^{-1} = (3 + 4i) \times (x + yi),$$

so that

$$1 = (3x - 4y) + (3y + 4x)i.$$

For equality, we must have both the real and imaginary parts equal, so we have the two equations

$$3x - 4y = 1,$$

$$4x + 3y = 0.$$

To solve these equations, add 3 times the first equation to 4 times the second. This gives

$$25x = 3,$$

hence $x = \frac{3}{25}$ and $y = -\frac{4}{25}$. So $z^{-1} = \frac{3}{25} - \frac{4}{25}i$. Thus the reciprocal of the complex number $z = 3 + 4i$ can be written as

$$z^{-1} = \frac{3 - 4i}{25}.$$

There is a pattern here. The top of the expression for z^{-1} is obtained from z by changing the sign of the imaginary part, while the bottom is the sum of the squares of both parts, since $25 = 3^2 + 4^2$. Does this pattern generalize?

Suppose that $z = a + bi$ and let

$$w = \frac{a - bi}{a^2 + b^2}.$$

Does w satisfy Equation (3.2), defining the reciprocal of z?

Exercise 3.6

Let $z = a + bi$ and $w = \dfrac{a - bi}{a^2 + b^2}$. Calculate $z \times w$ and show that it equals 1.

The result of Exercise 3.6 justifies the following definition.

Reciprocal of a complex number
If $z = a + bi$ $(z \neq 0)$, then its **reciprocal** is given by

$$z^{-1} = \frac{a - bi}{a^2 + b^2}.$$

Now that we know how to calculate reciprocals, we can perform division.

Division of complex numbers
If w and $z \neq 0$ are two complex numbers, then

$$\frac{w}{z} = w \times z^{-1}.$$

Suppose that $u \times z = w$, where z, u and w are complex numbers. So long as $z \neq 0$, we can rearrange this equation, just as we can with real numbers, to obtain $u = w/z$.

Exercise 3.7

Let $w = -3 + 4i$ and $z = 12 - 5i$. Find z^{-1} and hence solve the equation $u \times z = w$.

Neither subtraction nor division are, in general, commutative; that is,

$$z - w \neq w - z, \qquad \frac{w}{z} \neq \frac{z}{w}.$$

Though the formula for the reciprocal is not complicated, it is not essential to remember it. To find the reciprocal of a complex number $z = a + bi$, we write $z^{-1} = 1/(a + bi)$ and then multiply both top and bottom by $a - bi$. That is,

$$z^{-1} = \frac{1}{a + bi} = \frac{a - bi}{(a + bi) \times (a - bi)} = \frac{a - bi}{a^2 + b^2}.$$

Definition

If $z = a + bi$, then its **complex conjugate** is

$$\overline{z} = a - bi.$$

The complex conjugate of $\overline{z}$ is just z again. The symbol $\overline{z}$ is pronounced 'z bar'.

Complex conjugates appear in a variety of contexts. In particular, they can be used to simplify any complex 'fraction' (that is, make its denominator real) by multiplying top and bottom by the complex conjugate of each complex number in the denominator.

Exercise 3.8

(a) Let $z = a + bi$. Calculate $z \times \overline{z}$.

(b) Simplify the complex fraction

$$\frac{1+2i}{(1+i)\times(2-3i)}.$$

(c) Show that if a, b and c are real numbers ($a \neq 0$), the solutions of the quadratic equation $ax^2 + bx + c = 0$ are either real or a pair of complex conjugates.

3.3 Geometry of complex numbers

We can visualize real numbers as lying on the number line (see Figure 3.1). Is there a similar way of picturing complex numbers?

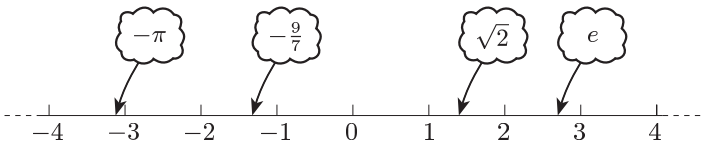


Figure 3.1 Number line

Argand diagram

A complex number is described by two real numbers, the real and imaginary parts, which suggests that we might use a point in the plane to represent it. So we represent the complex number $z = x + yi$ by the point (x, y) in the plane. Hence the complex numbers $3 + 4i$, $1 - 2i$ and $-3 + i$ are represented by the points $(3, 4)$, $(1, -2)$ and $(-3, 1)$, respectively (see Figure 3.2). Considered as complex numbers, the integers -1, 0, 1, 2 and 3 are represented by the points $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 0)$ and $(3, 0)$, while the complex numbers $-i$, i, $2i$ and $3i$ are represented by the points $(0, -1)$, $(0, 1)$, $(0, 2)$ and $(0, 3)$. In Figure 3.2, real numbers are shown on the horizontal axis, while points on the vertical axis represent multiples of i.

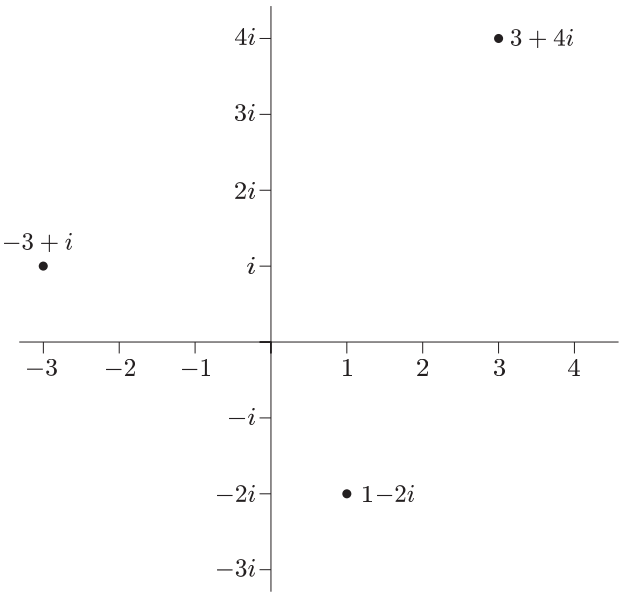


Figure 3.2 Some complex numbers visualized as points in the plane

A representation of complex numbers by points in a plane is usually referred to as an **Argand diagram**, after the French mathematician Jean-Robert Argand (1768–1822).

This representation may also be referred to as the **complex plane**.

> ***Definition***
>
> An **Argand diagram** provides a graphical representation of the complex number $x + yi$ as the point with Cartesian coordinates (x, y).

Exercise 3.9

Mark the complex numbers $-i$, $1 + 2i$ and $-1 - 2i$ on an Argand diagram.

How might we visualize addition of complex numbers on an Argand diagram? Suppose that $z = a + bi$ and $w = c + di$. Then, by definition,

$$z + w = (a + c) + (b + d)i.$$

So for the complex numbers represented by the points (a, b) and (c, d), their sum is represented by the point $(a + c, b + d)$. We have just added each component separately. This is illustrated in Figure 3.3.

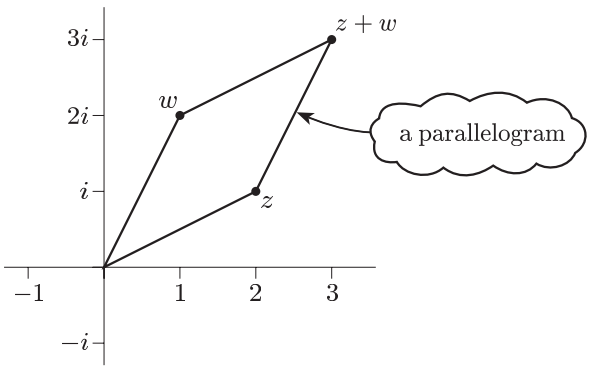


Figure 3.3 Sum $z + w = 3 + 3i$ of the complex numbers $z = 2 + i$ and $w = 1 + 2i$ shown on an Argand diagram

Exercise 3.10

Let $z = 2 + i$ and $w = 1 + 3i$.

(a) Mark z, $2z$ and $-3z$ on an Argand diagram and describe the geometric relationship between them.

(b) Mark z, w and $z - w$ on an Argand diagram and describe the geometric relationship between them.

Polar form

We now look at an alternative representation of complex numbers. This is based on polar (rather than Cartesian) coordinates and leads to a neat way of expressing multiplication of complex numbers.

The **polar coordinates** of a point are just another way of locating it in the plane (see Figure 3.4(a)). The position of a point P in the plane is represented by $\langle r, \theta \rangle$, where r is the distance OP and θ is the angle between the positive x-axis and the line OP, anticlockwise being taken as the positive direction. Here $r \geq 0$ and we can choose θ to be in the range $-\pi < \theta \leq \pi$. Other values of θ just give an alternative representation. For example, $\langle 1, 3\pi \rangle$ represents the same point as $\langle 1, \pi \rangle$. We shall use angle brackets to distinguish polar coordinates from Cartesian coordinates. The latter will be given in round brackets, as before.

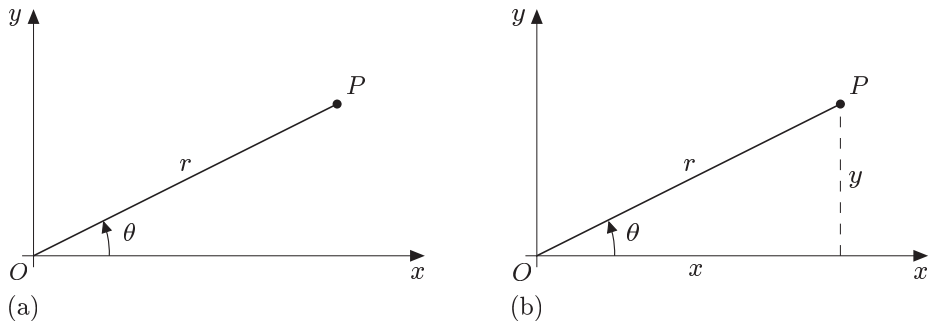


Figure 3.4 (a) Point P with polar coordinates $\langle r, \theta \rangle$ (b) Point P has polar coordinates $\langle r, \theta \rangle$ and Cartesian coordinates (x, y)

We have seen that a complex number $z = x + yi$ can be represented by the point in the plane with Cartesian coordinates (x, y). This point will also have a representation in polar coordinates, say $\langle r, \theta \rangle$. These polar coordinates give an alternative representation of the complex number z, which we call its **polar form**. We refer to the usual representation of z, written $x + yi$, as its **Cartesian form**.

Whenever there are two representations for a single mathematical idea we need to see how they are related. This is shown in Figure 3.4(b) and we can read off the relationships

These relationships also hold in the other three quadrants (quarters) of the plane.

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Hence the complex number with polar form $\langle r, \theta \rangle$ has Cartesian form

$$r\cos\theta + (r\sin\theta)i,$$

which we can also write as

$$r(\cos\theta + i\sin\theta).$$

Definition

The complex number $x + yi$ has **polar form** $\langle r, \theta \rangle$, where $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 3.11

Find the Cartesian form of the following complex numbers that are given in polar form.

(a) $\langle 1, 0\rangle$. (b) $\langle 1, \frac{\pi}{4}\rangle$. (c) $\langle 2, \frac{\pi}{2}\rangle$.

Conversely, suppose we have a complex number in Cartesian form $z = x + yi$ and we want to find the corresponding polar form. This time we want to find r and θ in terms of x and y. Look again at Figure 3.4(b). By Pythagoras's Theorem, we have $r^2 = x^2 + y^2$ and hence

$$r = \sqrt{x^2 + y^2}$$

(where we take the positive square root). We call this distance the **modulus** of the complex number, written $|z|$.

Definition

If $z = a + bi$, then its **modulus** is

$$|z| = \sqrt{a^2 + b^2}.$$

Once we have found the modulus r, we then find θ by solving the trigonometric equations

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}.$$

Notice that $\tan\theta = y/x$. We call this angle θ the **argument** of z and write it as $\arg(z)$.

Example 3.1

Suppose $z = -1 + i$. Find the modulus, argument and polar form of z.

Solution

We have $r^2 = (-1)^2 + 1^2$ and hence the modulus is $r = \sqrt{2}$. To find the argument we need to solve

$$\cos\theta = -\tfrac{1}{\sqrt{2}}, \qquad \sin\theta = \tfrac{1}{\sqrt{2}}.$$

To do this we look at an Argand diagram for z (Figure 3.5) and see that z lies in the second quadrant. Hence we expect a value for θ lying between $\frac{\pi}{2}$ and π.

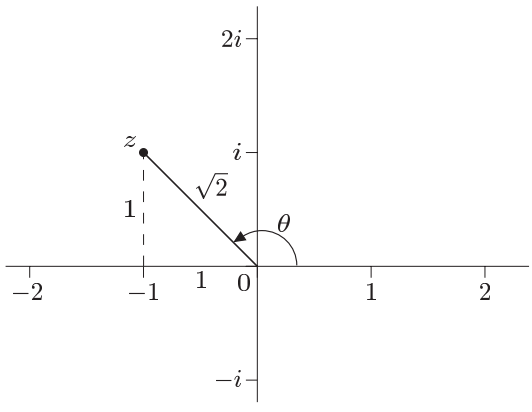


Figure 3.5 Argand diagram showing $-1 + i$

The angle θ in Figure 3.5 lies between $\frac{\pi}{2}$ and π with $\cos\theta = -\frac{1}{\sqrt{2}}$. So the argument θ of z is $\frac{3\pi}{4}$. Since $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$, the polar form for the complex number $-1 + i$ is $\langle\sqrt{2}, \frac{3\pi}{4}\rangle$. ■

In Example 3.1 we found a particular value for the argument θ. However, there is some ambiguity in this value. We could have ended up pointing in the same direction after completing any additional number of complete rotations around the origin in either direction. Hence the argument is not unique — it can differ by any multiple of 2π. This ambiguity extends to the polar form of any complex number and we have the following general result.

If the complex number z has polar forms $\langle r, \theta\rangle$ and $\langle s, \phi\rangle$, then

$$r = s, \qquad \theta - \phi = 2\pi m,$$

where m is an integer.

Exercise 3.12

Express in polar form the following complex numbers.

(a) i. (b) -2. (c) $-2 - 2i$.

The Argand diagram in Solution 3.12(c) shows the argument of $-2 - 2i$ as $\frac{5\pi}{4}$. This is one value. Of the possible values, the one that satisfies $-\pi < \theta \leq \pi$ is called the **principal value of the argument**. So the principal value of $\arg(-2 - 2i)$ is $-\frac{3\pi}{4}$. It is the principal value that is usually used when giving a complex number in polar form.

Complex multiplication in polar form

Using the polar form, it is possible to give a simple description of multiplication of complex numbers. Suppose that z has polar form $\langle r, \theta\rangle$ and w has polar form $\langle s, \phi\rangle$. What is the polar form of $z \times w$? To find this, we first go back into Cartesian form. We have

$$\begin{aligned} z \times w &= r(\cos\theta + i\sin\theta) \times s(\cos\phi + i\sin\phi) \\ &= rs\cos\theta\cos\phi + i(rs\cos\theta\sin\phi) + i(rs\sin\theta\cos\phi) \\ &\quad + i^2(rs\sin\theta\sin\phi) \\ &= rs((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)). \end{aligned}$$

To complete the analysis we need the trigonometric identities

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

If these identities are new to you, you may wish to use your calculator to test them.

So we have

$$z \times w = rs(\cos(\theta + \phi) + i\sin(\theta + \phi)).$$

This is the Cartesian form of the complex number with polar form

$$\langle rs, \theta + \phi\rangle.$$

Hence multiplying two complex numbers multiplies their moduli and adds their arguments, so we have an elegant description of the multiplication of

complex numbers in terms of their polar forms.

Multiplication of complex numbers in polar form
If $z = \langle r, \theta \rangle$ and $w = \langle s, \phi \rangle$, then

$$z \times w = \langle rs, \theta + \phi \rangle.$$

Exercise 3.13

Let $z = -1 + i$ and $w = -2 - 2i$. The polar forms of these are

$$z = \left\langle \sqrt{2}, \tfrac{3\pi}{4} \right\rangle, \qquad w = \left\langle 2\sqrt{2}, -\tfrac{3\pi}{4} \right\rangle.$$

Using these polar forms, calculate each of the following products, then express the results in Cartesian form.

(a) $z \times w$. (b) $z \times z$. (c) $z^3 = z \times (z \times z)$.

The polar form gives us a simple way of calculating powers of complex numbers. Suppose a complex number z has polar form $\langle r, \theta \rangle$. Then

$$z^2 = \langle r, \theta \rangle \times \langle r, \theta \rangle = \langle r \times r, \theta + \theta \rangle = \langle r^2, 2\theta \rangle,$$
$$z^3 = z^2 \times z = \langle r^2, 2\theta \rangle \times \langle r, \theta \rangle = \langle r^2 \times r, 2\theta + \theta \rangle = \langle r^3, 3\theta \rangle,$$
$$\vdots$$

As we increase the power, the modulus increases by the factor r while the argument increases by adding θ.

Powers of a complex number in polar form
For a complex number $z = \langle r, \theta \rangle$

$$z^n = \langle r^n, n\theta \rangle,$$

where n is an integer.

3.4 Complex exponentials

In this subsection we introduce a final operation on complex numbers. What might e^z mean, where z is a complex number?

Euler's formula

We start by looking at complex exponentials of the form $e^{i\theta}$.

One possible approach to the definition of certain functions is to use a series, for example, the Taylor series mentioned in Section 1. In particular, for e^x (with x in $\mathbb{R}$), we have

$$e^x = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots.$$

This is a definition of e^x.

Suppose we put $x = i\theta$ into this series. This suggests that

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \tfrac{1}{2!}(i\theta)^2 + \tfrac{1}{3!}(i\theta)^3 + \tfrac{1}{4!}(i\theta)^4 + \tfrac{1}{5!}(i\theta)^5 + \cdots \\ &= 1 + i\theta - \tfrac{1}{2!}\theta^2 - i\tfrac{1}{3!}\theta^3 + \tfrac{1}{4!}\theta^4 + i\tfrac{1}{5!}\theta^5 + \cdots. \end{aligned} \tag{3.3}$$

Looking at just the real terms in Equation (3.3), they are

$$1 - \tfrac{1}{2!}\theta^2 + \tfrac{1}{4!}\theta^4 - \tfrac{1}{6!}\theta^6 + \cdots .$$

This is the Taylor series for $\cos\theta$.

Looking at just the multiples of i in Equation (3.3), they are

$$i\left(\theta - \tfrac{1}{3!}\theta^3 + \tfrac{1}{5!}\theta^5 - \tfrac{1}{7!}\theta^7 + \cdots\right).$$

The terms in the brackets form the Taylor series for $\sin\theta$.

Together, these suggest that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.4}$$

The discussion leading to Equation (3.4) is certainly not a proof! It does not give the meaning of infinite series containing complex terms, nor the validity of separating and regrouping terms in an infinite series, as we have done. But it is a persuasive discussion and we shall use Equation (3.4) as a definition of $e^{i\theta}$. Equation (3.4) is known as **Euler's formula**.

Euler introduced this famous formula in 1748.

Euler's formula gives us a definition of e^z in the case where z is a real multiple of i. How might we deal with a general complex exponent, say $z = a + bi$? Well, we should expect that

$$e^{a+bi} = e^a \times e^{bi} = e^a(\cos b + i\sin b). \tag{3.5}$$

We take Equation (3.5) as our definition of e^z for a general complex exponent $z = a + bi$.

Definition

For the complex number $z = a + bi$,

$$e^{a+bi} = e^a \times e^{bi} = e^a(\cos b + i\sin b). \tag{3.5}$$

Exercise 3.14

Express e^{2+3i} in Cartesian form.

Now $e^{a+bi} = e^a(\cos b + i\sin b)$ has polar form $\langle e^a, b\rangle$. We can use this to verify that, with this definition, complex exponentials have some properties that we would expect, by analogy with real exponentials.

Exercise 3.15

Let $z = a + bi$ and $w = c + di$ be any two complex numbers.

(a) Give the following in polar form.

(i) $e^z \times e^w$. (ii) e^{z+w}.

Show that $e^z \times e^w = e^{z+w}$.

For real numbers x and y, $e^x \times e^y = e^{x+y}$.

(b) Express e^{-z} in polar form. Calculate $e^z \times e^{-z}$ and show that it is 1.

For any real number x, e^{-x} means $1/e^x$, so $e^x \times e^{-x} = 1$.

The expression on the right-hand side of Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

is already familiar from our discussion of the relationship between the polar and Cartesian forms for a complex number. We know that

$$\langle r, \theta\rangle = r(\cos\theta + i\sin\theta).$$

Comparing these two formulas we see that

$$\langle r, \theta \rangle = re^{i\theta}.$$

So our definition of the complex exponential gives us yet another representation of complex numbers, which we shall refer to as the *exponential form*.

> ***Definition***
>
> The complex number $x + yi$ has **exponential form** $re^{i\theta}$, where $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 3.16

Put the complex number $1 + i$ in polar form and then in exponential form.

We now have three different representations of complex numbers: Cartesian form, polar form and exponential form. Each is useful in its own way and each throws light on the others.

End-of-section Exercises

Exercise 3.17

If $z = 2 + i$ and $w = 1 - 2i$, find the following.

(a) $z + w$. (b) $-w$. (c) $z - w$. (d) $-2z$.

(e) $z \times w$. (f) w^{-1}. (g) $\dfrac{z}{w}$.

Exercise 3.18

Let $z = a + bi$ and $w = c + di$ be any two complex numbers. Show that

$$\overline{z} + \overline{w} = \overline{z + w}.$$

Exercise 3.19

Let $z = 2 - i$ and $w = \langle 2, \frac{\pi}{3} \rangle$.

(a) Show the points z and w on an Argand diagram.

(b) Convert z to polar form and w to Cartesian form.

(c) What is w^3 in Cartesian form?

Exercise 3.20

(a) For $z = 12 - 5i$, find the following.

(i) $\overline{z}$. (ii) $|z|$. (iii) $z \times \overline{z}$.

(b) For a general complex number z, show that $z \times \overline{z} = |z|^2$.

(c) For a general complex number z, describe $1/z$ in terms of its conjugate $\overline{z}$ and modulus $|z|$.

Exercise 3.21

Give $e^{-1+i\pi/3}$ in Cartesian form.

Solutions to the exercises

Section 1

1.1 In each case, entries in Table 1.1 are used.

(a) From Table 1.1,
$f'(x) = 4e^{4x}$.

(b) By the Constant Multiple Rule,
$g'(t) = 6 \times 3t^2 = 18t^2$.

(c) By the Constant Multiple Rule,
$\dfrac{ds}{dt} = 4 \times 3\cos(3t) = 12\cos(3t)$.

(d) By the Constant Multiple and Sum Rules,
$\dfrac{dy}{dx} = 7 \times 4x^3 - 3 \times 2x + 4 \times 0 = 28x^3 - 6x$.

(e) By the Constant Multiple and Sum Rules,
$k'(x) = 6e^x + \dfrac{1}{x}$.

1.2 In each case, entries in Table 1.1 are used.

(a) By the Constant Multiple Rule,
$f''(x) = 4 \times 4e^{4x} = 16e^{4x}$.

(b) By the Constant Multiple Rule,
$g''(t) = 18 \times 2t = 36t$.

(c) By the Constant Multiple Rule,
$\dfrac{d^2s}{dt^2} = 12 \times (-3\sin(3t)) = -36\sin(3t)$.

(d) By the Constant Multiple and Sum Rules,
$\dfrac{d^2y}{dx^2} = 84x^2 - 6$.

(e) By the Constant Multiple and Sum Rules,
$k''(x) = 6e^x + (-1)x^{-2} = 6e^x - \dfrac{1}{x^2}$.

1.3 In each case, the Product Rule is used. The Sum and Constant Multiple Rules are also used where appropriate. A full solution is given only for (a).

(a) Let $f(x) = x$ and $g(x) = \cos x$. Then
$k(x) = f(x)g(x)$.
Using the Product Rule, we have
$$\begin{aligned} k'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= 1 \times \cos x + x \times (-\sin x) \\ &= \cos x - x\sin x. \end{aligned}$$

(b) $$\begin{aligned} k'(x) &= \cos x \cos x + \sin x(-\sin x) \\ &= \cos^2 x - \sin^2 x. \end{aligned}$$

(c) $$\begin{aligned} k'(t) &= 3\cos(3t)\cos(4t) + \sin(3t)(-4\sin(4t)) \\ &= 3\cos(3t)\cos(4t) - 4\sin(3t)\sin(4t) \end{aligned}$$

(d) $$\begin{aligned} k'(t) &= (6t+2)e^t + (3t^2 + 2t + 1)e^t \\ &= (3t^2 + 8t + 3)e^t. \end{aligned}$$

(e) $$\begin{aligned} \frac{dy}{dx} &= e^x \sin(2x) + e^x(2\cos(2x)) \\ &= e^x(\sin(2x) + 2\cos(2x)). \end{aligned}$$

(f) $$\begin{aligned} \frac{dy}{dx} &= 2x\cos(2x) + (x^2+2)(-2\sin(2x)) \\ &= 2x\cos(2x) - 2(x^2+2)\sin(2x). \end{aligned}$$

(g) $$\begin{aligned} f'(x) &= \cos x \sin x + \sin x \cos x \\ &= 2\cos x \sin x. \end{aligned}$$

(h) $$\begin{aligned} f'(x) &= 4x\sin x + 2x^2\cos x \\ &\quad + 2x\cos x + (x^2+1)(-\sin x) \\ &= 2x(x+1)\cos x \\ &\quad - (x^2 - 4x + 1)\sin x. \end{aligned}$$

1.4 Each case uses the Quotient Rule. In addition, the Sum Rule is used in (b) and (e), and the Constant Multiple Rule is used in (b), (e) and (f).

(a) Let $f(x) = \sin x$ and $g(x) = x$. Then
$k(x) = f(x)/g(x)$, $f'(x) = \cos x$ and $g'(x) = 1$.
Using the Quotient Rule, we have
$$\begin{aligned} k'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\ &= \frac{(\cos x)x - \sin x \times 1}{x^2} \\ &= \frac{x\cos x - \sin x}{x^2}. \end{aligned}$$

(b) Let $f(x) = 2x$ and $g(x) = 1 + x$. Then
$k(x) = f(x)/g(x)$, $f'(x) = 2$ and $g'(x) = 1$.
Using the Quotient Rule, we have
$$\begin{aligned} k'(x) &= \frac{2 \times (1+x) - 2x \times 1}{(1+x)^2} \\ &= \frac{2}{(1+x)^2}. \end{aligned}$$

(c) Let $f(x) = \sin x$ and $g(x) = \cos x$. Then
$k(x) = f(x)/g(x)$, $f'(x) = \cos x$ and $g'(x) = -\sin x$.
Using the Quotient Rule, we have
$$\begin{aligned} k'(x) &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x}, \end{aligned}$$
since $\cos^2 x + \sin^2 x = 1$.

(d) Let $f(t) = t$ and $g(t) = \sin t$. Then
$k(t) = f(t)/g(t)$, $f'(t) = 1$ and $g'(t) = \cos t$.
Using the Quotient Rule, we have
$$k'(t) = \frac{\sin t - t\cos t}{\sin^2 t}.$$

(e) Using the Quotient Rule, we have

$$k'(x) = \frac{6x(1-x) - (3x^2-1)(-1)}{(1-x)^2}$$
$$= \frac{-3x^2+6x-1}{(1-x)^2}.$$

(f) Using the Quotient Rule, we have

$$\frac{dy}{dx} = \frac{(-\sin x)2x^2 - (\cos x)4x}{(2x^2)^2}$$
$$= -\frac{2x(x\sin x + 2\cos x)}{4x^4}$$
$$= -\frac{x\sin x + 2\cos x}{2x^3}.$$

(g) Using the Quotient Rule, we have

$$\frac{ds}{dt} = \frac{2e^{2t}\sin(3t) - e^{2t}3\cos(3t)}{\sin^2(3t)}$$
$$= \frac{(2\sin(3t) - 3\cos(3t))e^{2t}}{\sin^2(3t)}.$$

1.5 **(a)** $k(x) = \sec x = 1/\cos x$.

Using the Quotient Rule, we have

$$k'(x) = \frac{0\times\cos x - 1\times(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \frac{\sin x}{\cos x}\times\frac{1}{\cos x}$$
$$= \tan x\sec x.$$

(b) $k(x) = \operatorname{cosec} x = 1/\sin x$.

Using the Quotient Rule, we have

$$k'(x) = \frac{0\times\sin x - 1\times\cos x}{\sin^2 x}$$
$$= -\frac{\cos x}{\sin^2 x}$$
$$= -\frac{\cos x}{\sin x}\times\frac{1}{\sin x}$$
$$= -\cot x\operatorname{cosec} x.$$

(c) $k(x) = \cot x = 1/\tan x = \cos x/\sin x$.

Using the Quotient Rule, we have

$$k'(x) = \frac{(-\sin x)\sin x - \cos x\cos x}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\operatorname{cosec}^2 x.$$

1.6 The solutions alternate between using the Chain Rule and the Composite Rule.

(a) Let $y = \sin(ax)$ and $u = ax$. Then $y = \sin u$, $dy/du = \cos u$ and $du/dx = a$.

Using the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= (\cos u)a$$
$$= a\cos(ax).$$

(b) Let $k(x) = \cos(ax)$, $f(x) = \cos x$ and $g(x) = ax$. Then $k(x) = f(g(x))$. Since $f'(x) = -\sin x$ and $g(x) = ax$, $f'(g(x)) = -\sin(ax)$. Also, $g'(x) = a$.

Using the Composite Rule, we have

$$k'(x) = f'(g(x))g'(x)$$
$$= (-\sin(ax))a$$
$$= -a\sin(ax).$$

(c) Let $y = e^{ax}$ and $u = ax$. Then $y = e^u$, $dy/du = e^u$ and $du/dx = a$.

Using the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= e^u a$$
$$= ae^{ax}.$$

(d) Let $k(x) = \ln(ax)$, $f(x) = \ln x$ and $g(x) = ax$. Then $k(x) = f(g(x))$. Since $f'(x) = 1/x$ and $g(x) = ax$, $f'(g(x)) = 1/(ax)$. Also $g'(x) = a$.

Using the Composite Rule, we have

$$k'(x) = f'(g(x))g'(x)$$
$$= \frac{1}{ax}\times a$$
$$= \frac{1}{x}.$$

1.7 The solutions alternate between using function notation and Leibniz notation.

(a) Let $k(x) = \cos(x^4)$, $f(x) = \cos x$ and $g(x) = x^4$. Then $k(x) = f(g(x))$. Since $f'(x) = -\sin x$ and $g(x) = x^4$, $f'(g(x)) = -\sin(x^4)$. Also, $g'(x) = 4x^3$.

Using the Composite Rule, we have

$$k'(x) = f'(g(x))g'(x)$$
$$= (-\sin(x^4))4x^3$$
$$= -4x^3\sin(x^4).$$

(b) Let $y = e^{\operatorname{cosec} x}$ and $u = \operatorname{cosec} x$. Then $y = e^u$, $dy/du = e^u$ and $du/dx = -\cot x\operatorname{cosec} x$.

Using the Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= e^u(-\cot x\operatorname{cosec} x)$$
$$= -\cot x\operatorname{cosec} x(e^{\operatorname{cosec} x}).$$

(c) Let $k(x) = \cos^2 x$, $f(x) = x^2$ and $g(x) = \cos x$. Then $k(x) = f(g(x))$. Since $f'(x) = 2x$ and $g(x) = \cos x$, $f'(g(x)) = 2\cos x$. Also $g'(x) = -\sin x$.

Using the Composite Rule, we have

$$\begin{aligned} k'(x) &= f'(g(x))g'(x) \\ &= 2\cos x(-\sin x) \\ &= -2\cos x \sin x. \end{aligned}$$

(d) Let $y = \ln(1 + x^6)$ and $u = 1 + x^6$. Then $y = \ln u$, $dy/du = 1/u$ and $du/dx = 6x^5$.

Using the Chain Rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \frac{1}{u} \times 6x^5 \\ &= \frac{6x^5}{1 + x^6}. \end{aligned}$$

(e) Let $k(t) = (t^3 + 2)^5$, $f(t) = t^5$ and $g(t) = t^3 + 2$. Then $k(t) = f(g(t))$. Since $f'(t) = 5t^4$ and $g(x) = t^3 + 2$, $f'(g(x)) = 5(t^3 + 2)^4$. Also, $g'(t) = 3t^2$.

Using the Composite Rule, we have

$$\begin{aligned} k'(t) &= f'(g(t))g'(t) \\ &= 5(t^3 + 2)^4 3t^2 \\ &= 15t^2(t^3 + 2)^4. \end{aligned}$$

(f) Let $s = \cot(t^2 - 3t + 1)$ and $u = t^2 - 3t + 1$. Then $s = \cot u$, $ds/du = -\operatorname{cosec}^2 u$ and $du/dt = 2t - 3$.

Using the Chain Rule, we have

$$\begin{aligned} \frac{ds}{dt} &= \frac{ds}{du} \times \frac{du}{dt} \\ &= -\operatorname{cosec}^2 u \times (2t - 3) \\ &= -(2t - 3)\operatorname{cosec}^2(t^2 - 3t + 1). \end{aligned}$$

(g) Let $k(x) = \sqrt{x^3 + 4x + 3} = (x^3 + 4x + 3)^{\frac{1}{2}}$, $f(x) = x^{\frac{1}{2}}$ and $g(x) = x^3 + 4x + 3$. Then $k(x) = f(g(x))$. Since $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $g(x) = x^3 + 4x + 3$, $f'(g(x)) = \frac{1}{2}(x^3 + 4x + 3)^{-\frac{1}{2}}$. Also, $g'(x) = 3x^2 + 4$.

Using the Composite Rule, we have

$$\begin{aligned} k'(x) &= f'(g(x))g'(x) \\ &= \tfrac{1}{2}(x^3 + 4x + 3)^{-\frac{1}{2}}(3x^2 + 4) \\ &= \frac{3x^2 + 4}{2\sqrt{x^3 + 4x + 3}}. \end{aligned}$$

1.8 **(a)** Here we have a product in which the second term is a composite. Let $h(x) = x$ and $k(x) = \ln(1 + x + x^2)$. Then $f(x) = h(x)k(x)$.

Using the Product Rule, we have

$$f'(x) = h'(x)k(x) + h(x)k'(x). \tag{S.1}$$

Now $h'(x) = 1$ and we shall find $k'(x)$ using the Composite Rule. Let $p(x) = \ln x$ and $q(x) = 1 + x + x^2$. Then $k(x) = p(q(x))$. Since $p'(x) = 1/x$ and $q(x) = 1 + x + x^2$, $p'(q(x)) = 1/(1 + x + x^2)$. Also, $q'(x) = 1 + 2x$.

Using the Composite Rule, we have

$$\begin{aligned} k'(x) &= p'(q(x))q'(x) \\ &= \frac{1}{1 + x + x^2} \times (1 + 2x) \\ &= \frac{1 + 2x}{1 + x + x^2}. \end{aligned}$$

Substituting this into (S.1), we have

$$\begin{aligned} f'(x) &= 1 \times \ln(1 + x + x^2) + x \times \frac{1 + 2x}{1 + x + x^2} \\ &= \ln(1 + x + x^2) + \frac{x(1 + 2x)}{1 + x + x^2}. \end{aligned}$$

(b) This is a product of three terms. Let $g(x) = x^3 \tan x$. Then $f(x) = g(x)\ln x$.

Using the Product Rule, we have

$$g'(x) = 3x^2 \tan x + x^3 \sec^2 x.$$

Then, using the Product Rule again, we have

$$\begin{aligned} f'(x) &= (3x^2 \tan x + x^3 \sec^2 x)\ln x + (x^3 \tan x)\frac{1}{x} \\ &= (3x^2 \tan x + x^3 \sec^2 x)\ln x + x^2 \tan x. \end{aligned}$$

(c) This is a composite where the inner function is a product. We use Leibniz notation.

Let $w = (x + 4)\sec x$, $u = x + 4$ and $v = \sec x$. Then $y = \cos w$, $w = uv$, $du/dx = 1$ and $dv/dx = \tan x \sec x$.

Using the Product Rule, we have

$$\begin{aligned} \frac{dw}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= \sec x + (x + 4)\tan x \sec x. \end{aligned}$$

We also have

$$\frac{dy}{dw} = -\sin w.$$

Using the Chain Rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dw} \times \frac{dw}{dx} \\ &= -\sin w((x + 4)\tan x \sec x + \sec x) \\ &= -((x + 4)\tan x \sec x + \sec x)\sin((x + 4)\sec x) \\ &= -\sec x((x + 4)\tan x + 1)\sin((x + 4)\sec x). \end{aligned}$$

(d) This is a composite of a composite. It is probably easier to use Leibniz notation. We work from the innermost argument.

Let $u = \sin x$ and $v = \sin u$. Then $y = \sin v$, $du/dx = \cos x$, $dv/du = \cos u$ and $dy/dv = \cos v$.

Using the Chain Rule, we have

$$\frac{dv}{dx} = \frac{dv}{du} \times \frac{du}{dx}, \tag{S.2}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{dx}. \tag{S.3}$$

Substituting (S.2) into (S.3), we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx} \\ &= \cos v \times \cos u \times \cos x. \end{aligned}$$

Substituting for u and v, and noting that $v = \sin u = \sin(\sin x)$, we obtain

$$\frac{dy}{dx} = \cos(\sin(\sin x))\cos(\sin x)\cos x.$$

1.9 The distance of the object from the fixed point is given by $s = e^{2t}\cos(\pi t)$.

Using the Product Rule, we have

$$\begin{aligned}\frac{ds}{dt} &= 2e^{2t}\cos(\pi t) + e^{2t}(-\pi\sin(\pi t)) \\ &= 2e^{2t}\cos(\pi t) - \pi e^{2t}\sin(\pi t) \\ &= e^{2t}(2\cos(\pi t) - \pi\sin(\pi t)),\end{aligned}$$

which gives the velocity.

Using the Product Rule again, we have

$$\begin{aligned}\frac{d^2s}{dt^2} &= 2e^{2t}(2\cos(\pi t) - \pi\sin(\pi t)) \\ &\quad + e^{2t}(-2\pi\sin(\pi t) - \pi^2\cos(\pi t)) \\ &= e^{2t}(4\cos(\pi t) - 2\pi\sin(\pi t) \\ &\quad - 2\pi\sin(\pi t) - \pi^2\cos(\pi t)) \\ &= e^{2t}((4-\pi^2)\cos(\pi t) - 4\pi\sin(\pi t)),\end{aligned}$$

which gives the acceleration.

When $t = 1$, we have

$$\begin{aligned}\frac{ds}{dt} &= e^2(2\cos\pi - \pi\sin\pi) \\ &= -2e^2,\end{aligned}$$

since $\cos\pi = -1$ and $\sin\pi = 0$. Also,

$$\begin{aligned}\frac{d^2s}{dt^2} &= e^2((4-\pi^2)\cos\pi - 4\pi\sin\pi) \\ &= -(4-\pi^2)e^2 \\ &= (\pi^2-4)e^2.\end{aligned}$$

Thus, one second after the start of the motion, the velocity is $-2e^2$, i.e. $-14.8\,\mathrm{m\,s^{-1}}$ (to 1 d.p.), and the acceleration is $(\pi^2-4)e^2$, i.e. $43.4\,\mathrm{m\,s^{-2}}$ (to 1 d.p.).

1.10 Using the Product Rule, we have

$$\begin{aligned}f'(x) &= 1\times e^{-x} + x(-e^{-x}) \\ &= e^{-x} - xe^{-x} \\ &= (1-x)e^{-x}. \qquad \text{(S.4)}\end{aligned}$$

Since e^{-x} is never zero, the equation

$$f'(x) = (1-x)e^{-x} = 0$$

has only one solution, $x = 1$.

When $x = 1$, $f(x) = e^{-1}$, so $(1, e^{-1})$ is a stationary point.

Differentiating (S.4) using the Product Rule, we have

$$\begin{aligned}f''(x) &= -e^{-x} + (1-x)(-e^{-x}) \\ &= (x-2)e^{-x}.\end{aligned}$$

Hence $f''(1) = -e^{-1} < 0$.

Since $f''(1)$ is negative, by the Second Derivative Test, the point $(1, e^{-1})$ is a local maximum.

1.11 Using the Quotient Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x(1-x) - x^2(-1)}{(1-x)^2} \\ &= \frac{2x - 2x^2 + x^2}{(1-x)^2} \\ &= \frac{2x - x^2}{(1-x)^2} \\ &= \frac{x(2-x)}{(1-x)^2}.\end{aligned}$$

For horizontal tangents, we require $dy/dx = 0$, and $dy/dx = 0$ when $x(2-x) = 0$, i.e. when $x = 0$ and $x = 2$. When $x = 0$, $y = 0$ and when $x = 2$, $y = 4/(-1) = -4$. So the points at which the tangent is horizontal are $(0, 0)$ and $(2, -4)$.

1.12 Let $u = -x^2$. Then $y = e^u$, $dy/du = e^u$ and $du/dx = -2x$.

Using the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx} \\ &= e^u(-2x) \\ &= -2xe^{x^2}.\end{aligned}$$

At a stationary point, $dy/dx = 0$, i.e. $-2xe^{-x^2} = 0$, so $x = 0$. When $x = 0$, $y = e^0 = 1$, so the stationary point is $(0, 1)$.

Using the Product Rule, we have

$$\begin{aligned}\frac{d^2y}{dx^2} &= -2e^{-x^2} + (-2x)(-2xe^{-x^2}) \\ &= (4x^2-2)e^{-x^2}.\end{aligned}$$

When $x = 0$, $d^2y/dx^2 = -2$. This is negative, so, by the Second Derivative Test, the stationary point is a local maximum.

1.13 **(a)** Let $y = \arcsin(x^3)$ and $u = x^3$. Then $y = \arcsin u$, $dy/du = 1/\sqrt{1-u^2}$ and $du/dx = 3x^2$.

Using the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx} \\ &= \frac{1}{\sqrt{1-u^2}}\times 3x^2 \\ &= \frac{3x^2}{\sqrt{1-(x^3)^2}} \\ &= \frac{3x^2}{\sqrt{1-x^6}}.\end{aligned}$$

(b) Let $y = \ln(-x)$ and $u = -x$, where $x < 0$ and hence $u > 0$. Then $y = \ln u$, $dy/dx = 1/u$ and $du/dx = -1$.

Using the Chain Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\times\frac{du}{dx} \\ &= \frac{1}{u}(-1) = -\frac{1}{(-x)} = \frac{1}{x},\end{aligned}$$

so that

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{x} \quad (x < 0).$$

1.14 **(a)** In this case, $f(x) = \sin x$, so $f'(x) = \cos x$. Thus $f(0) = 0$ and $f'(0) = 1$, so

$$\begin{aligned} p(x) &= f(0) + (x-0)f'(0) \\ &= 0 + (x-0)1 \\ &= x. \end{aligned}$$

(b) If $f(x) = \cos x$, $f'(x) = -\sin x$. Thus $f(0) = 1$, and $f'(0) = 0$, so

$$\begin{aligned} p(x) &= f(0) + (x-0)f'(0) \\ &= 1 + (x-0)0 \\ &= 1. \end{aligned}$$

(c) Here $f(x) = e^x$ and so $f'(x) = e^x$. Thus $f(1) = e^1$ and $f'(1) = e^1$, so

$$\begin{aligned} p(x) &= f(1) + (x-1)f'(1) \\ &= e^1 + (x-1)e^1 \\ &= ex. \end{aligned}$$

(d) $f(x) = x^3 + 2x - 5$, so $f'(x) = 3x^2 + 2$. Thus $f(1) = -2$ and $f'(1) = 5$, so

$$\begin{aligned} p(x) &= f(1) + (x-1)f'(1) \\ &= -2 + (x-1)5 \\ &= 5x - 7. \end{aligned}$$

1.15 **(a)** The values of $f(x) = \cos x$ and its derivatives at $x = 0$ are

$$f(x) = \cos x, \quad f(0) = 1,$$
$$f'(x) = -\sin x, \quad f'(0) = 0,$$
$$f''(x) = -\cos x, \quad f''(0) = -1.$$

Hence

$$p_2(x) = 1 - \tfrac{1}{2}x^2.$$

(b) The values of $f(x) = \sin x$ and its derivatives at $x = 0$ are

$$f(x) = \sin x, \quad f(0) = 0,$$
$$f'(x) = \cos x, \quad f'(0) = 1,$$
$$f''(x) = -\sin x, \quad f''(0) = 0.$$

Hence

$$p_2(x) = x.$$

(c) The values of $f(x) = \ln(1+x)$ and its derivatives at $x = 0$ are

$$f(x) = \ln(1+x), \quad f(0) = 0,$$
$$f'(x) = 1/(1+x), \quad f'(0) = 1,$$
$$f''(x) = -1/(1+x)^2, \quad f''(0) = -1.$$

Hence

$$p_2(x) = x - \tfrac{1}{2}x^2.$$

(d) Working to four decimal places, the values of $f(x) = \sqrt{1+x}$ and its derivatives at $x = 0.2$ are

$$f(x) = \sqrt{1+x}, \quad f(0.2) = 1.0954,$$
$$f'(x) = 1/(2\sqrt{1+x}), \quad f'(0.2) = 0.4564,$$
$$f''(x) = -1/(4(1+x)^{3/2}), \quad f''(0.2) = -0.1902.$$

Hence

$$p_2(x) = 1.0954 + 0.4564(x - 0.2) - \tfrac{0.1902}{2}(x-0.2)^2.$$

1.16 We have

$$f(x) = \ln(1+x), \quad f(0) = 0,$$
$$f'(x) = 1/(1+x), \quad f'(0) = 1,$$
$$f''(x) = -1/(1+x)^2, \quad f''(0) = -1,$$
$$f'''(x) = 2/(1+x)^3, \quad f'''(0) = 2.$$

Hence the cubic Taylor polynomial is

$$p_3(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3.$$

1.17 **(a)** The values of $f(x) = \sqrt{1+x}$ and its derivatives at $x = 0$ are

$$f(x) = \sqrt{1+x}, \quad f(0) = 1,$$
$$f'(x) = \tfrac{1}{2}(1+x)^{-\frac{1}{2}}, \quad f'(0) = \tfrac{1}{2},$$
$$f''(x) = -\tfrac{1}{4}(1+x)^{-\frac{3}{2}}, \quad f''(0) = -\tfrac{1}{4},$$
$$f'''(x) = \tfrac{3}{8}(1+x)^{-\frac{5}{2}}, \quad f'''(0) = \tfrac{3}{8},$$
$$f^{(4)}(x) = -\tfrac{15}{16}(1+x)^{-\frac{7}{2}}, \quad f^{(4)}(0) = -\tfrac{15}{16},$$
$$f^{(5)}(x) = \tfrac{105}{32}(1+x)^{-\frac{9}{2}}, \quad f^{(5)}(0) = \tfrac{105}{32}.$$

Hence

$$p_5(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \tfrac{5}{128}x^4 + \tfrac{7}{256}x^5$$

(b) Using (a), we have

$$\begin{aligned} \sqrt{1+\tfrac{1}{9}} &\simeq 1 + \tfrac{1}{2}\left(\tfrac{1}{9}\right) - \tfrac{1}{8}\left(\tfrac{1}{9}\right)^2 + \tfrac{1}{16}\left(\tfrac{1}{9}\right)^3 \\ &\quad - \tfrac{5}{128}\left(\tfrac{1}{9}\right)^4 + \tfrac{7}{256}\left(\tfrac{1}{9}\right)^5 \\ &= 1 + 0.055\,556 - 0.001\,543 + 0.000\,086 \\ &\quad - 0.000\,006 + 0.000\,000 \\ &\simeq 1.054\,093. \end{aligned}$$

(The fact that $\frac{7}{256}\left(\frac{1}{9}\right)^5 = 0.000\,000$ to six decimal places shows that we can be reasonably confident that $\sqrt{1+\frac{1}{9}} \simeq 1.054\,093$ is accurate to four decimal places.)

Therefore

$$\begin{aligned} \sqrt{10} = \sqrt{9\left(1+\tfrac{1}{9}\right)} &= 3\sqrt{1+\tfrac{1}{9}} \\ &\simeq 3 \times 1.054\,093 \\ &= 3.162\,279 \\ &\simeq 3.1623 \quad \text{(to 4 d.p.)}. \end{aligned}$$

1.18 We give only brief solutions here.

(a) The function $f(x) = \sin(e^x)$ is a composite.

Using the Composite Rule, we have

$$f'(x) = \cos(e^x)e^x.$$

(b) The function $f(x) = \cos^2(3x)$ is a composite of a composite.

Using the Composite Rule twice, we have

$$\begin{aligned} f'(x) &= 2\cos(3x) \times (-\sin(3x)) \times 3 \\ &= -6\cos(3x)\sin(3x). \end{aligned}$$

(c) The function $g(y) = \ln(\cos y)/y$ is a quotient, the numerator being a composite.

Using the Quotient Rule and the Composite Rule,

$$g'(y) = \frac{\left(\frac{1}{\cos y} \times (-\sin y)\right) y - \ln(\cos y) \times 1}{y^2}$$
$$= -\frac{y \tan y + \ln(\cos y)}{y^2}.$$

(d) The function $k(t) = \sin(t^2)\cos t$ is a product in which one of the functions is a composite.

Using the Product Rule and the Composite Rule,

$$k'(t) = (\cos(t^2) \times 2t)\cos t + \sin(t^2)(-\sin t)$$
$$= 2t\cos t\cos(t^2) - \sin t \sin(t^2).$$

(e) The function given by

$$y = 2\sin(t+1) - t\cos(t^2-1)$$

involves sums, a constant multiple, a product and two composites.

Using the corresponding rules, we have

$$\frac{dy}{dt} = 2\cos(t+1) \times 1$$
$$- (1 \times \cos(t^2-1) + t(-\sin(t^2-1) \times 2t))$$
$$= 2\cos(t+1) - \cos(t^2-1) + 2t^2\sin(t^2-1).$$

1.19 Let $V(x) = x\sqrt{9-x^2}$ $(x > 0)$. This function is a product in which one function is a composite. Let $f(x) = \sqrt{9-x^2} = (9-x^2)^{\frac{1}{2}}$. Then $V(x) = xf(x)$.

We have to show that $V(x) = 4.5$ is a local maximum and find the value of x.

Using the Product Rule, we have

$$V'(x) = f(x) + xf'(x)$$
$$= \sqrt{9-x^2} + xf'(x).$$

To find $f'(x)$, we use the Composite Rule:

$$f'(x) = \tfrac{1}{2}(9-x^2)^{-\frac{1}{2}}(-2x)$$
$$= \frac{-x}{\sqrt{9-x^2}}.$$

Then

$$V'(x) = \sqrt{9-x^2} + x\left(\frac{-x}{\sqrt{9-x^2}}\right)$$
$$= \frac{(\sqrt{9-x^2})^2 - x^2}{\sqrt{9-x^2}}$$
$$= \frac{9-x^2-x^2}{\sqrt{9-x^2}}$$
$$= \frac{9-2x^2}{\sqrt{9-x^2}}. \qquad \text{(S.5)}$$

Now $V'(x) = 0$ when $9 - 2x^2 = 0$, i.e. when $x^2 = \frac{9}{2}$. Thus $x = \frac{3}{\sqrt{2}}$ (we are interested only in positive values of x). So the only stationary point of V for $x > 0$ is at $x = \frac{3}{\sqrt{2}}$. We next use the Second Derivative Test to show that this stationary point is a local maximum.

We use the Quotient Rule to find $V''(x)$. From (S.5),

$$V'(x) = \frac{9-2x^2}{\sqrt{9-x^2}}$$
$$= \frac{9-2x^2}{f(x)}$$

and we have already found the derivative of $f(x)$.

Using the Quotient Rule, we have

$$V''(x) = \frac{-4xf(x) - (9-2x^2)f'(x)}{(f(x))^2}.$$

We require the value of $V''(x)$ when $x = \frac{3}{\sqrt{2}}$, at which point $9 - 2x^2 = 0$. So

$$V''\left(\tfrac{3}{\sqrt{2}}\right) = \frac{-4 \times \frac{3}{\sqrt{2}}}{f(\frac{3}{\sqrt{2}})} = -\frac{4 \times \frac{3}{\sqrt{2}}}{\sqrt{9-\frac{9}{2}}},$$

which is negative. Hence, by the Second Derivative Test, $x = \frac{3}{\sqrt{2}}$ gives a local maximum.

Finally, we check that $V(\frac{3}{\sqrt{2}}) = 4.5$, as required.

$$V(\tfrac{3}{\sqrt{2}}) = \tfrac{3}{\sqrt{2}}\sqrt{9-\tfrac{9}{2}}$$
$$= \tfrac{3}{\sqrt{2}}\sqrt{\tfrac{9}{2}}$$
$$= \tfrac{3}{\sqrt{2}} \times \tfrac{3}{\sqrt{2}}$$
$$= 4.5.$$

1.20 **(a)** If $f(x) = e^{x^2}$, then $f(1) = e^1$ and

$$f'(x) = 2xe^{x^2}, \quad f'(1) = 2e^1,$$
$$f''(x) = (2+4x^2)e^{x^2}, \quad f''(1) = 6e^1,$$
$$f'''(x) = 4x(3+2x^2)e^{x^2}, \quad f'''(1) = 20e^1.$$

Thus the required polynomials are

$$p_1(x) = f(1) + (x-1)f'(1)$$
$$= e^1 + 2e^1(x-1)$$
$$= e + 2e(x-1),$$
$$p_2(x) = f(1) + (x-1)f'(1) + \tfrac{1}{2}(x-1)^2 f''(1)$$
$$= e^1 + 2e^1(x-1) + 3e^1(x-1)^2$$
$$= e + 2e(x-1) + 3e(x-1)^2,$$
$$p_3(x) = f(1) + (x-1)f'(1) + \tfrac{1}{2}(x-1)^2 f''(1)$$
$$+ \tfrac{1}{3!}(x-1)^3 f'''(1)$$
$$= e^1 + 2e^1(x-1) + 3e^1(x-1)^2 + \tfrac{10}{3}e^1(x-1)^3$$
$$= e + 2e(x-1) + 3e(x-1)^2 + \tfrac{10}{3}e(x-1)^3.$$

To compute the next polynomial, you only need to add in the additional term — as we shall do from now on.

(b) If $f(x) = \dfrac{1}{2-x}$, then $f(1) = 1$ and

$$f'(x) = \frac{1}{(2-x)^2}, \quad f'(1) = 1,$$

$$f''(x) = \frac{2}{(2-x)^3}, \quad f''(1) = 2,$$

$$f'''(x) = \frac{6}{(2-x)^4}, \quad f'''(1) = 6.$$

Thus the required polynomials are

$$\begin{aligned} p_1(x) &= f(1) + (x-1)f'(1) \\ &= 1 + (x-1), \\ p_2(x) &= 1 + (x-1) + \tfrac{1}{2}(x-1)^2 f''(1) \\ &= 1 + (x-1) + (x-1)^2, \\ p_3(x) &= 1 + (x-1) + (x-1)^2 \\ &\quad + \tfrac{1}{3!}(x-1)^3 f'''(1) \\ &= 1 + (x-1) + (x-1)^2 + (x-1)^3. \end{aligned}$$

(c) If $f(x) = \ln x$, then $f(1) = 0$ and

$$f'(x) = \frac{1}{x}, \quad f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2}, \quad f''(1) = -1,$$

$$f'''(x) = \frac{2}{x^3}, \quad f'''(1) = 2.$$

Thus the required polynomials are

$$\begin{aligned} p_1(x) &= f(1) + (x-1)f'(1) \\ &= x - 1, \\ p_2(x) &= (x-1) + \tfrac{1}{2}(x-1)^2 f''(1) \\ &= (x-1) + -\tfrac{1}{2}(x-1)^2, \\ p_3(x) &= (x-1) - \tfrac{1}{2}(x-1)^2 \\ &\quad + \tfrac{1}{3!}(x-1)^3 f'''(1) \\ &= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3. \end{aligned}$$

Section 2

2.1 In each case, c is an arbitrary constant.

(a) $\displaystyle\int (5x + x^3)\,dx = \tfrac{5}{2}x^2 + \tfrac{1}{4}x^4 + c.$

(b) $$\begin{aligned} \int \frac{1+\sqrt{x}}{x}\,dx &= \int \left(\frac{1}{x} + x^{-\frac{1}{2}}\right) dx \\ &= \ln x + 2x^{\frac{1}{2}} + c. \end{aligned}$$

(c) $\displaystyle\int e^{7u}\,du = \tfrac{1}{7}e^{7u} + c.$

(d) $\displaystyle\int \sin(2t) - 3\cos(3t)\,dt = -\tfrac{1}{2}\cos(2t) - \sin(3t) + c.$

(e) As $\cos(2x) = 2\cos^2 x - 1$, $\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$, so we have

$$\begin{aligned} \int \cos^2 x\,dx &= \int \tfrac{1}{2}(\cos(2x) + 1)\,dx \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}\sin(2x) + x\right) + c \\ &= \tfrac{1}{4}(\sin(2x) + 2x) + c. \end{aligned}$$

2.2 **(a)** $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + c$

(b) $\displaystyle\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + c$

(c) $\displaystyle\int \operatorname{cosec}^2 t\,dt = -\cot t + c$

2.3 **(a)** We have

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-2x}{(1+x^2)^2}.$$

(b) Using (a), we have

$$\begin{aligned} \int \frac{x}{(1+x^2)^2}\,dx &= -\tfrac{1}{2}\int \frac{-2x}{(1+x^2)^2}\,dx \\ &= \frac{-1}{2(1+x^2)} + c. \end{aligned}$$

2.4 **(a)** $$\begin{aligned} \int_0^1 \sin(\pi x)\,dx &= -\tfrac{1}{\pi}\left[\cos(\pi x)\right]_0^1 \\ &= -\tfrac{1}{\pi}[\cos\pi - \cos 0] \\ &= \tfrac{2}{\pi} \\ &= 0.64 \quad \text{(to 2 d.p.)}. \end{aligned}$$

(b) $$\begin{aligned} \int_{-1}^3 e^{4x} + x^3\,dx &= \left[\tfrac{1}{4}e^{4x} + \tfrac{1}{4}x^4\right]_{-1}^3 \\ &= \tfrac{1}{4}(e^{12} + 81 - e^{-4} - 1) \\ &= 20 + \tfrac{1}{4}(e^{12} - e^{-4}) \\ &= 40\,708.69 \quad \text{(to 2 d.p.)}. \end{aligned}$$

(c) $$\begin{aligned} \int_{-\pi/4}^{\pi/4} \sec^2 t\,dt &= \left[\tan t\right]_{-\pi/4}^{\pi/4} \\ &= \tan\tfrac{\pi}{4} - \tan(-\tfrac{\pi}{4}) \\ &= 2. \end{aligned}$$

(d) $$\begin{aligned} \int_{-1}^1 \frac{1}{1+u^2}\,du &= \left[\arctan u\right]_{-1}^1 \\ &= \tfrac{\pi}{4} - (-\tfrac{\pi}{4}) \\ &= \tfrac{\pi}{2} \\ &= 1.57 \quad \text{(to 2 d.p.)}. \end{aligned}$$

(e) $$\begin{aligned} \int_0^1 e^x\,dx &= \left[e^x\right]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \\ &= 1.72 \quad \text{(to 2 d.p.)}. \end{aligned}$$

2.5 In each case, let $F(x)$ be any integral of $f(x)$ and apply (2.8).

(a) $\displaystyle\int_a^a f(x)\,dx = F(a) - F(a) = 0.$

(b) Starting from the expression on the left-hand side, we have

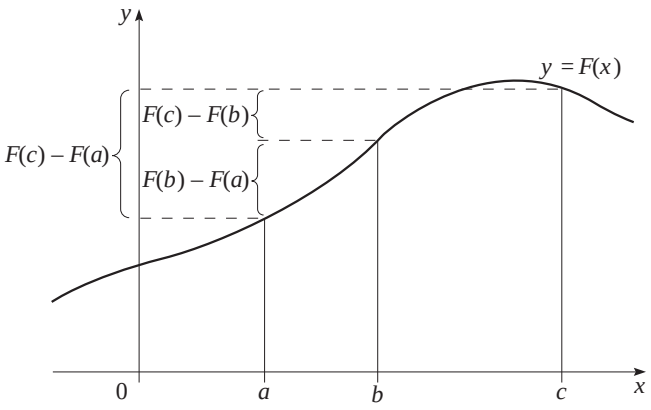
$$\begin{aligned} \int_b^a f(x)\,dx &= F(a) - F(b) \\ &= -(F(b) - F(a)) \\ &= -\int_a^b f(x)\,dx. \end{aligned}$$

In words, this says that the change in the value of any integral of $f(x)$ in moving from $x = b$ to $x = a$ is the negative of the change obtained by moving in the opposite direction.

(c) Starting from the left-hand side, we obtain

$$\begin{aligned}&\int_a^b f(x)\,dx + \int_b^c f(x)\,dx\\ &= F(b) - F(a) + F(c) - F(b)\\ &= F(c) - F(a)\\ &= \int_a^c f(x)\,dx.\end{aligned}$$

This result is illustrated below, where we have $F(c) - F(a) = (F(c) - F(b)) + (F(b) - F(a))$. If $F(x)$ is an integral of $f(x)$, each difference represents a definite integral of $f(x)$.



2.6 **(a)** The area is

$$\begin{aligned}\int_0^4 e^{-\frac{1}{2}x}\,dx &= \Big[-2e^{-\frac{1}{2}x}\Big]_0^4\\ &= -2(e^{-2} - e^0)\\ &= 1.7293 \quad \text{(to 4 d.p.)}.\end{aligned}$$

(b) The area is

$$\begin{aligned}\int_0^{\pi/6} \cos(3x)\,dx &= \big[\tfrac{1}{3}\sin(3x)\big]_0^{\pi/6}\\ &= \tfrac{1}{3}\big(\sin\tfrac{\pi}{2} - \sin 0\big)\\ &= \tfrac{1}{3}.\end{aligned}$$

(c) The x-axis is the line $y = 0$ and $y = 1 - x^2 = 0$ when $x = \pm 1$. We have $y = 1 - x^2 \geq 0$ when $x \in [-1, 1]$, so the required area is given by

$$\begin{aligned}\int_{-1}^1 1 - x^2\,dx &= \big[x - \tfrac{1}{3}x^3\big]_{-1}^1\\ &= \big(1 - \tfrac{1}{3}\big) - \big(-1 - \big(-\tfrac{1}{3}\big)\big)\\ &= \tfrac{4}{3}.\end{aligned}$$

(d) We have $x^2 + y^2 = 1$, so $y^2 = 1 - x^2$. Taking the (positive) square root, gives $y = \sqrt{1 - x^2}$. The graph of this function cuts the positive x-axis at $x = 1$ and the positive y-axis at $x = 0$, so the required integral is

$$\int_0^1 \sqrt{1 - x^2}\,dx.$$

2.7 The graph of $y = \cos(3x)$ is positive when $x \in \big[0, \frac{\pi}{6}\big)$ and negative for $x \in \big(\frac{\pi}{6}, \frac{\pi}{2}\big)$. It is zero at $x = \frac{\pi}{6}$ and at $x = \frac{\pi}{2}$. So the required area is

$$\int_0^{\pi/6} \cos(3x)\,dx - \int_{\pi/6}^{\pi/2} \cos(3x)\,dx.$$

Using the value obtained in Solution 2.6(b) for the first integral, we have

$$\begin{aligned}\tfrac{1}{3} - \big[\tfrac{1}{3}\sin(3x)\big]_{\pi/6}^{\pi/2} &= \tfrac{1}{3} - \tfrac{1}{3}(-1 - 1)\\ &= \tfrac{1}{3} + \tfrac{2}{3} = 1.\end{aligned}$$

2.8 We use the formula for integration by parts:

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

(a) Let $f(x) = x$ and $g'(x) = \cos x$, so $f'(x) = 1$ and $g(x) = \sin x$. Then

$$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx\\ &= x\sin x - (-\cos x) + c\\ &= x\sin x + \cos x + c,\end{aligned}$$

as we saw on page 32.

(b) Let $f(x) = x$ and $g'(x) = e^x$, so $f'(x) = 1$ and $g(x) = e^x$. Then

$$\begin{aligned}\int xe^x\,dx &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + c\\ &= e^x(x - 1) + c.\end{aligned}$$

2.9 **(a)** Let $f(x) = \ln x$ and $g'(x) = x$, so $f'(x) = 1/x$ and $g(x) = \frac{1}{2}x^2$. Then

$$\begin{aligned}\int x\ln x\,dx &= \tfrac{1}{2}x^2\ln x - \int \frac{1}{x} \times \tfrac{1}{2}x^2\,dx\\ &= \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,dx\\ &= \tfrac{1}{4}x^2(2\ln x - 1) + c.\end{aligned}$$

(b) Using the solution to (a) gives

$$\begin{aligned}\int_1^2 x\ln x\,dx &= \big[\tfrac{1}{4}x^2(2\ln x - 1)\big]_1^2\\ &= (2\ln 2 - 1) - \tfrac{1}{4}(2\ln 1 - 1)\\ &= 2\ln 2 - 1 + \tfrac{1}{4}\\ &= 0.6363 \quad \text{(to 4 d.p.)}.\end{aligned}$$

2.10 **(a)** Let $f(x) = x$ and $g'(x) = \cos(2x)$, so $f'(x) = 1$ and $g(x) = \frac{1}{2}\sin(2x)$. Then

$$\begin{aligned}\int x\cos(2x)\,dx &= \tfrac{1}{2}x\sin(2x) - \int \tfrac{1}{2}\sin(2x)\,dx\\ &= \tfrac{1}{2}x\sin(2x) + \tfrac{1}{4}\cos(2x) + c.\end{aligned}$$

(b) Let $f(x) = x$ and $g'(x) = \sin(3x)$, so $f'(x) = 1$ and $g(x) = -\frac{1}{3}\cos(3x)$. Then

$$\begin{aligned}\int x\sin(3x)\,dx &= -\tfrac{1}{3}x\cos(3x) + \int \tfrac{1}{3}\cos(3x)\,dx\\ &= -\tfrac{1}{3}x\cos(3x) + \tfrac{1}{9}\sin(3x) + c\\ &= \tfrac{1}{9}(\sin(3x) - 3x\cos(3x)) + c.\end{aligned}$$

2.11 **(a)** Let $f(x) = x$ and $g'(x) = e^{2x}$, so $f'(x) = 1$ and $g(x) = \frac{1}{2}e^{2x}$. Then

$$\begin{aligned}\int xe^{2x}\,dx &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + c \\ &= \tfrac{1}{4}e^{2x}(2x-1) + c.\end{aligned}$$

(b) We have $e^{2x} > 0$ for all values of x, and $x > 0$ if $x \in [1, 2]$, so the product xe^{2x} is positive on this interval. Hence its graph lies above the x-axis.

The area under the graph between $x = 1$ and $x = 2$ is given by $\int_1^2 xe^{2x}\,dx$.

(c) The area is given by

$$\begin{aligned}\int_1^2 xe^{2x}\,dx &= \left[\tfrac{1}{4}e^{2x}(2x-1)\right]_1^2 \\ &= \tfrac{1}{4}e^4(4-1) - \tfrac{1}{4}e^2(2-1) \\ &= \tfrac{1}{4}(3e^4 - e^2) \\ &= 39.1013 \quad \text{(to 4 d.p.)}.\end{aligned}$$

2.12 Let $f(x) = x$ and $g'(x) = e^{3x}$, so $f'(x) = 1$ and $g(x) = \frac{1}{3}e^{3x}$. Then

$$\begin{aligned}\int_0^1 xe^{3x}\,dx &= \left[\tfrac{1}{3}xe^{3x}\right]_0^1 - \tfrac{1}{3}\int_0^1 e^{3x}\,dx \\ &= \tfrac{1}{3}e^3 - \left[\tfrac{1}{9}e^{3x}\right]_0^1 \\ &= \tfrac{1}{3}e^3 - \left(\tfrac{1}{9}e^3 - \tfrac{1}{9}e^0\right) \\ &= \tfrac{1}{9}(2e^3 + 1) \\ &= 4.5746 \quad \text{(to 4 d.p.)}.\end{aligned}$$

2.13 **(a)** We have $\sin\theta > 0$ if $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. On putting $\theta = 3x$, we have $\sin(3x) > 0$ if $x \in \left[\frac{\pi}{12}, \frac{\pi}{6}\right]$.

Multiplying a positive value of $\sin(3x)$ by x, which is itself positive in this interval, gives a positive value for the expression $x\sin(3x)$. So its graph lies above the x-axis for the whole of the interval $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$.

The area under the graph of $y = x\sin(3x)$ between $x = \frac{\pi}{12}$ and $x = \frac{\pi}{6}$ is given by

$$\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx.$$

(b) Let $f(x) = x$ and $g'(x) = \sin(3x)$, so $f'(x) = 1$ and $g(x) = -\frac{1}{3}\cos(3x)$. Then the area is given by

$$\begin{aligned}&\int_{\pi/12}^{\pi/6} x\sin(3x)\,dx \\ &= \left[-\tfrac{1}{3}x\cos(3x)\right]_{\pi/12}^{\pi/6} + \tfrac{1}{3}\int_{\pi/12}^{\pi/6}\cos(3x)\,dx \\ &= \left(0 + \tfrac{\pi}{36}\cos\tfrac{\pi}{4}\right) + \tfrac{1}{9}\left[\sin(3x)\right]_{\pi/12}^{\pi/6} \\ &= \tfrac{\pi}{36}\times\tfrac{1}{\sqrt{2}} + \tfrac{1}{9}\left(1 - \tfrac{1}{\sqrt{2}}\right) \\ &= 0.0943 \quad \text{(to 4 d.p.)}.\end{aligned}$$

2.14 **(a)** Take $u = 5 + 2x^2$. Then $du/dx = 4x$, so

$$\begin{aligned}\int (5+2x^2)^{16}x\,dx &= \int \tfrac{1}{4}(5+2x^2)^{16}(4x)\,dx \\ &= \tfrac{1}{4}\int u^{16}\,du \\ &= \tfrac{1}{4}\times\tfrac{1}{17}u^{17} + c \\ &= \tfrac{1}{68}(5+2x^2)^{17} + c.\end{aligned}$$

(b) Take $u = x^3$. Then $du/dx = 3x^2$, so

$$\begin{aligned}\int x^2\sec^2(x^3)\,dx &= \int \tfrac{1}{3}\sec^2(x^3)(3x^2)\,dx \\ &= \tfrac{1}{3}\int \sec^2 u\,du \\ &= \tfrac{1}{3}\tan u + c \\ &= \tfrac{1}{3}\tan(x^3) + c.\end{aligned}$$

(c) Take $u = \cos x$. Then $du/dx = -\sin x$, so

$$\begin{aligned}\int (\sin x)e^{\cos x}\,dx &= \int -e^{\cos x}(-\sin x)\,dx \\ &= -\int e^u\,du \\ &= -e^u + c \\ &= -e^{\cos x} + c.\end{aligned}$$

(d) Take $u = 3x$. Then $du/dx = 3$, so

$$\begin{aligned}\int \frac{1}{1+9x^2}\,dx &= \int \tfrac{1}{3}\left(\frac{1}{1+9x^2}\right)(3)\,dx \\ &= \tfrac{1}{3}\int \frac{1}{1+u^2}\,du \\ &= \tfrac{1}{3}\arctan u + c \\ &= \tfrac{1}{3}\arctan(3x) + c.\end{aligned}$$

2.15 Take $u = 1 + x^2$. Then $du/dx = 2x$, so

$$\begin{aligned}\int \frac{x}{1+x^2}\,dx &= \tfrac{1}{2}\int \frac{1}{1+x^2}(2x)\,dx \\ &= \tfrac{1}{2}\int \frac{1}{u}\,du \\ &= \tfrac{1}{2}\ln u + c \\ &= \tfrac{1}{2}\ln(1+x^2) + c.\end{aligned}$$

2.16 **(a)** Take $u = 6 + x^3$. Then $du/dx = 3x^2$, so

$$\begin{aligned}\int \left(6+x^3\right)^9 x^2\,dx &= \int \tfrac{1}{3}\left(6+x^3\right)^9(3x^2)\,dx \\ &= \tfrac{1}{3}\int u^9\,du \\ &= \tfrac{1}{3}\times\tfrac{1}{10}u^{10} + c \\ &= \tfrac{1}{30}(6+x^3)^{10} + c.\end{aligned}$$

(b) Take $u = e^{2x}$. Then $du/dx = 2e^{2x}$, so

$$\begin{aligned}\int e^{2x}\sin(e^{2x})\,dx &= \int \tfrac{1}{2}\sin(e^{2x})(2e^{2x})\,dx \\ &= \tfrac{1}{2}\int \sin u\,du \\ &= -\tfrac{1}{2}\cos u + c \\ &= -\tfrac{1}{2}\cos(e^{2x}) + c.\end{aligned}$$

(c) Take $u = \sin x$. Then $du/dx = \cos x$, so

$$\begin{aligned}\int \sin^4 x(\cos x)\,dx &= \int u^4\,du \\ &= \tfrac{1}{5}u^5 + c \\ &= \tfrac{1}{5}\sin^5 x + c.\end{aligned}$$

(d) Take $u = 1 + x^2$. Then $du/dx = 2x$, so

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx &= \int \tfrac{1}{2}\sqrt{1+x^2}\,(2x)\,dx \\ &= \tfrac{1}{2}\int \sqrt{u}\,du \\ &= \tfrac{1}{2}\times\tfrac{2}{3}u^{\frac{3}{2}} + c \\ &= \tfrac{1}{3}(1+x^2)^{\frac{3}{2}} + c.\end{aligned}$$

(e) Take $u = 1 + x^3$. Then $du/dx = 3x^2$, so

$$\begin{aligned}\int \frac{x^2}{1+x^3}\,dx &= \int \tfrac{1}{3}\left(\frac{1}{1+x^3}\right)(3x^2)\,dx \\ &= \tfrac{1}{3}\int \frac{1}{u}\,du \\ &= \tfrac{1}{3}\ln u + c \\ &= \tfrac{1}{3}\ln(1+x^3) + c.\end{aligned}$$

2.17 **(a)** Take $u = 1 + 2x^2$. Then $du/dx = 4x$. We have $u = 1$ when $x = 0$ and $u = 3$ when $x = 1$. So

$$\begin{aligned}\int_0^1 \frac{x}{1+2x^2}\,dx &= \int_0^1 \tfrac{1}{4}\left(\frac{1}{1+2x^2}\right)(4x)\,dx \\ &= \tfrac{1}{4}\int_1^3 \frac{1}{u}\,du \\ &= \tfrac{1}{4}\left[\ln u\right]_1^3 \\ &= \tfrac{1}{4}(\ln 3 - \ln 1) = \tfrac{1}{4}\ln 3.\end{aligned}$$

(b) Take $u = 1 + \cos^2 x$. Then $du/dx = -2\cos x \sin x$. We have $u = 2$ when $x = 0$ and $u = 1$ when $x = \frac{\pi}{2}$. So

$$\begin{aligned}&\int_0^{\pi/2} \frac{\sin x\cos x}{1+\cos^2 x}\,dx \\ &= \int_0^{\pi/2} -\tfrac{1}{2}\left(\frac{1}{1+\cos^2 x}\right)(-2\cos x\sin x)\,dx \\ &= -\tfrac{1}{2}\int_2^1 \frac{1}{u}\,du \\ &= \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du \\ &= \tfrac{1}{2}[\ln u]_1^2 \\ &= \tfrac{1}{2}[\ln 2 - \ln 1] = \tfrac{1}{2}\ln 2.\end{aligned}$$

(c) Take $u = -3x^2 - 2$. Then $du/dx = -6x$. We have $u = -5$ when $x = -1$ and $u = -2$ when $x = 0$. So

$$\begin{aligned}\int_{-1}^0 xe^{-3x^2-2}\,dx &= \int_{-1}^0 -\tfrac{1}{6}\left(e^{-3x^2-2}\right)(-6x)\,dx \\ &= -\tfrac{1}{6}\int_{-5}^{-2} e^u\,du \\ &= -\tfrac{1}{6}\left[e^u\right]_{-5}^{-2} \\ &= \tfrac{1}{6}\left(e^{-5} - e^{-2}\right) \\ &= -0.0214 \quad \text{(to 4 d.p.)}.\end{aligned}$$

(d) Take $u = \sec x$. Then $du/dx = \tan x \sec x$. We have $u = 1$ when $x = 0$ and $u = \sqrt{2}$ when $x = \frac{\pi}{4}$. So

$$\begin{aligned}\int_0^{\pi/4} \sec^3 x\tan x\,dx &= \int_0^{\pi/4} \sec^2 x(\tan x\sec x)\,dx \\ &= \int_1^{\sqrt{2}} u^2\,du \\ &= \left[\tfrac{1}{3}u^3\right]_1^{\sqrt{2}} \\ &= \tfrac{1}{3}(2\sqrt{2} - 1) \\ &= 0.6095 \quad \text{(to 4 d.p.)}.\end{aligned}$$

2.18 **(a)** $\frac{1}{19}x^{19} + \frac{1}{2}x^2 + c.$

(b) $\dfrac{2}{x^3} + c.$

(c) $-3\cos\left(\frac{1}{3}x\right) + c.$

(d) $-\frac{1}{2}e^{-2x} + c.$

(e) $\tan x + c.$

(f) We have

$$\begin{aligned}&\frac{(3x^2+1)(x-1)}{\sqrt{x}} \\ &= \frac{3x^3 - 3x^2 + x - 1}{\sqrt{x}} \\ &= 3x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}} - x^{-\frac{1}{2}}.\end{aligned}$$

So the required integral is

$$\begin{aligned}&\int 3x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}} - x^{-\frac{1}{2}}\,dx \\ &= \tfrac{6}{7}x^{\frac{7}{2}} - \tfrac{6}{5}x^{\frac{5}{2}} + \tfrac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c.\end{aligned}$$

2.19 **(a)** $\frac{7}{9}\left[x^9\right]_0^1 = \frac{7}{9}(1-0) = \frac{7}{9}.$

(b) $\frac{1}{2}\left[\sin(2x)\right]_0^{\pi/4} = \frac{1}{2}(1-0) = \frac{1}{2}.$

(c) $\frac{1}{7}\left[e^{7x}\right]_{-1}^1 = \frac{1}{7}\left(e^7 - e^{-7}\right).$

(d)

$$\begin{aligned}-\left[\cot x\right]_{\pi/4}^{\pi/3} &= -\left(\cot\tfrac{\pi}{3} - \cot\tfrac{\pi}{4}\right) \\ &= -\left(\tfrac{1}{\sqrt{3}} - 1\right) \\ &= 1 - \tfrac{1}{\sqrt{3}}.\end{aligned}$$

(e) We have

$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$.

So that

$$\begin{aligned}\int_0^{\pi/4} \sin^2 x\,dx &= \tfrac{1}{2}\int_0^{\pi/4} 1 - \cos(2x)\,dx \\ &= \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin(2x)\right]_0^{\pi/4} \\ &= \tfrac{1}{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) \\ &= \tfrac{\pi}{8} - \tfrac{1}{4}.\end{aligned}$$

2.20 **(a)** The graph lies on or above the x-axis for $x \in [-1, 0]$, so the area is

$$\begin{aligned}\int_{-1}^0 x^2 - 4x\,dx &= \left[\tfrac{1}{3}x^3 - 2x^2\right]_{-1}^0 \\ &= (0-0) - \left(-\tfrac{1}{3} - 2\right) \\ &= \tfrac{7}{3}.\end{aligned}$$

(b) The graph lies on or below the x-axis for $x \in [0,2]$, so the area is

$$\begin{aligned} -\int_0^2 x^2 - 4x\,dx &= -\left[\tfrac{1}{3}x^3 - 2x^2\right]_0^2 \\ &= -\left(\tfrac{8}{3} - 8\right) + (0-0) \\ &= \tfrac{16}{3}. \end{aligned}$$

(c) Integrals only represent areas when the graph concerned is non-negative throughout the interval of integration, which is not so here. In this case, the total area is equal to $\frac{7}{3} + \frac{16}{3} = \frac{23}{3}$, whereas the integral is equal to

$$\begin{aligned} \int_{-1}^0 x^2 - 4x\,dx + \int_0^2 x^2 - 4x\,dx &= \tfrac{7}{3} - \tfrac{16}{3} \\ &= -3. \end{aligned}$$

2.21 **(a)** Let $f(x) = x$ and $g'(x) = e^{-x}$, so $f'(x) = 1$ and $g(x) = -e^{-x}$. Substitution into the integration by parts formula (2.9) gives

$$\begin{aligned} \int xe^{-x}\,dx &= -xe^{-x} - \int -e^{-x}\,dx \\ &= -xe^{-x} - e^{-x} + c \\ &= -e^{-x}(x+1) + c, \end{aligned}$$

where c is an arbitrary constant.

(b) Let $f(x) = \ln x$ and $g'(x) = \sqrt{x} = x^{\frac{1}{2}}$, so $f'(x) = 1/x$ and $g(x) = \frac{2}{3}x^{\frac{3}{2}}$. Then

$$\begin{aligned} \int \sqrt{x}\ln x\,dx &= (\ln x)\tfrac{2}{3}x^{\frac{3}{2}} - \int \frac{1}{x} \times \tfrac{2}{3}x^{\frac{3}{2}}\,dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln x - \tfrac{2}{3}\int x^{\frac{1}{2}}\,dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln x - \tfrac{4}{9}x^{\frac{3}{2}} + c, \end{aligned}$$

where c is an arbitrary constant.

(c) Let $f(x) = x$ and $g'(x) = \cos(4x)$ so $f'(x) = 1$ and $g(x) = \frac{1}{4}\sin(4x)$. Then

$$\begin{aligned} \int x\cos(4x)\,dx &= \tfrac{1}{4}x\sin(4x) - \tfrac{1}{4}\int \sin(4x)\,dx \\ &= \tfrac{1}{4}x\sin(4x) + \tfrac{1}{16}\cos(4x) + c, \end{aligned}$$

where c is an arbitrary constant.

(d) Using the result of (a), we have

$$\begin{aligned} \int_0^1 xe^{-x}\,dx &= \left[-e^{-x}(x+1)\right]_0^1 \\ &= -2e^{-1} + e^0 \\ &= 1 - 2e^{-1}. \end{aligned}$$

2.22 **(a)** Let $u = 1 + x^2$. Then $du/dx = 2x$, so

$$\begin{aligned} \int x(1+x^2)^3\,dx &= \tfrac{1}{2}\int (1+x^2)^3(2x)\,dx \\ &= \tfrac{1}{2}\int u^3\,du \\ &= \tfrac{1}{8}u^4 + c \\ &= \tfrac{1}{8}(1+x^2)^4 + c. \end{aligned}$$

(b) Let $u = \cos x$. Then $du/dx = -\sin x$, so

$$\begin{aligned} \int \sin x\cos^3 x\,dx &= -\int \cos^3 x\,(-\sin x)\,dx \\ &= -\int u^3\,du \\ &= -\tfrac{1}{4}u^4 + c \\ &= -\tfrac{1}{4}\cos^4 x + c. \end{aligned}$$

(c) Let $u = x^2 - 4x + 3$. Then $du/dx = 2x - 4$, so

$$\begin{aligned} \int \frac{x-2}{x^2-4x+3}\,dx &= \tfrac{1}{2}\int \frac{1}{x^2-4x+3}(2x-4)\,dx \\ &= \tfrac{1}{2}\int \frac{1}{u}\,du \quad (u>0) \\ &= \tfrac{1}{2}\ln u + c \\ &= \tfrac{1}{2}\ln(x^2-4x+3) + c. \end{aligned}$$

(d) Let $u = 1 + x^3$. Then $du/dx = 3x^2$. We have $x^3 = u - 1$. So

$$\begin{aligned} \int \frac{x^5}{1+x^3}\,dx &= \tfrac{1}{3}\int \frac{x^3}{1+x^3}(3x^2)\,dx \\ &= \tfrac{1}{3}\int \frac{u-1}{u}\,du \\ &= \tfrac{1}{3}\int 1 - \frac{1}{u}\,du \quad (u>0) \\ &= \tfrac{1}{3}u - \tfrac{1}{3}\ln u + c \\ &= \tfrac{1}{3}(1+x^3) - \tfrac{1}{3}\ln(1+x^3) + c. \end{aligned}$$

2.23 **(a)** Let $u = 1 + x^2$. Then $du/dx = 2x$, so

$$\begin{aligned} \int x(1+x^2)^5\,dx &= \tfrac{1}{2}\int (1+x^2)^5(2x)\,dx \\ &= \tfrac{1}{2}\int u^5\,du \\ &= \tfrac{1}{12}u^6 + c \\ &= \tfrac{1}{12}(1+x^2)^6 + c. \end{aligned}$$

(b) Let $u = \sin x$. Then $du/dx = \cos x$, so

$$\begin{aligned} \int \cos x\sin^3 x\,dx &= \int \sin^3 x\,(\cos x)\,dx \\ &= \int u^3\,du \\ &= \tfrac{1}{4}u^4 + c \\ &= \tfrac{1}{4}\sin^4 x + c. \end{aligned}$$

(c) Let $u = 4x^2 + 3x + 5$. Then $du/dx = 8x + 3$, so

$$\begin{aligned} &\int (24x+9)(4x^2+3x+5)^7\,dx \\ &\quad = 3\int (4x^2+3x+5)^7(8x+3)\,dx \\ &\quad = 3\int u^7\,du \\ &\quad = \tfrac{3}{8}u^8 + c \\ &\quad = \tfrac{3}{8}(4x^2+3x+5)^8 + c. \end{aligned}$$

(d) Let $u = 3x^2 + 4x - 1$. Then $du/dx = 6x + 4$, so

$$\begin{aligned}\int \frac{6x+4}{\sqrt{3x^2+4x-1}}\,dx &= \int \frac{1}{\sqrt{3x^2+4x-1}}(6x+4)\,dx \\ &= \int u^{-\frac{1}{2}}\,du \\ &= 2u^{\frac{1}{2}} + c \\ &= 2\sqrt{3x^2+4x-1} + c.\end{aligned}$$

2.24 **(a)** Let $u = \sqrt{x-2} = (x-2)^{\frac{1}{2}}$. Then $du/dx = \frac{1}{2}(x-2)^{-\frac{1}{2}}$ and $x = u^2 + 2$, so

$$\begin{aligned}\int \frac{x}{\sqrt{x-2}}\,dx &= 2\int x\left(\tfrac{1}{2}(x-2)^{-\frac{1}{2}}\right)\,dx \\ &= 2\int u^2 + 2\,du \\ &= 2\left(\tfrac{1}{3}u^3 + 2u\right) + c \\ &= \tfrac{2}{3}(x-2)^{\frac{3}{2}} + 4(x-2)^{\frac{1}{2}} + c.\end{aligned}$$

(b) Let $u = \sqrt{x} - 1 = x^{\frac{1}{2}} - 1$. Then $du/dx = \frac{1}{2}x^{-\frac{1}{2}}$. Also $u = 1$ when $x = 4$, and $u = 2$ when $x = 9$. So

$$\begin{aligned}\int_4^9 \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,dx &= 2\int_4^9 \frac{1}{\sqrt{x}-1}\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)\,dx \\ &= 2\int_1^2 \frac{1}{u}\,du \\ &= 2[\ln u]_1^2 \\ &= 2\ln 2.\end{aligned}$$

(c) Let $u = \arcsin\left(\frac{1}{3}x\right)$. Then

$$\frac{du}{dx} = \tfrac{1}{3} \times \frac{1}{\sqrt{1-(\frac{1}{3}x)^2}} = \frac{1}{\sqrt{9-x^2}},$$

so that

$$\begin{aligned}\int \frac{1}{\sqrt{9-x^2}}\,dx &= \int 1\,du \\ &= u + c \\ &= \arcsin\left(\tfrac{1}{3}x\right) + c.\end{aligned}$$

(d) Let $u = \arctan\left(\frac{1}{3}x\right)$. Then

$$\frac{du}{dx} = \tfrac{1}{3} \times \frac{1}{1+(\frac{1}{3}x)^2} = \frac{3}{9+x^2},$$

so that

$$\begin{aligned}\int \frac{1}{9+x^2}\,dx &= \tfrac{1}{3}\int \frac{3}{9+x^2}\,dx \\ &= \tfrac{1}{3}\int 1\,du = \tfrac{1}{3}u + c \\ &= \tfrac{1}{3}\arctan\left(\tfrac{1}{3}x\right) + c.\end{aligned}$$

2.25 **(a)** Let $u = \tan x$. Then $du/dx = \sec^2 x$, so

$$\begin{aligned}\int \frac{\sec^2 x}{\tan x}\,dx &= \int \frac{1}{\tan x}(\sec^2 x)\,dx \\ &= \int \frac{1}{u}\,du \quad (u > 0) \\ &= \ln u + c \\ &= \ln(\tan x) + c.\end{aligned}$$

(b) Let $u = 1 + x^2$. Then $du/dx = 2x$ and $x^2 = u - 1$, so that

$$\begin{aligned}\int \frac{x^3}{(1+x^2)^{\frac{1}{2}}}\,dx &= \tfrac{1}{2}\int \frac{x^2}{(1+x^2)^{\frac{1}{2}}}(2x)\,dx \\ &= \tfrac{1}{2}\int \frac{u-1}{u^{\frac{1}{2}}}\,du \\ &= \tfrac{1}{2}\int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du \\ &= \tfrac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + c \\ &= \tfrac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} - \left(1+x^2\right)^{\frac{1}{2}} + c.\end{aligned}$$

(c) Let $u = \arctan\left(\frac{2}{3}x\right)$. Then

$$\frac{du}{dx} = \tfrac{2}{3} \times \frac{1}{1+\left(\frac{2}{3}x\right)^2} = \frac{6}{9+4x^2},$$

so that

$$\begin{aligned}\int \frac{1}{4x^2+9}\,dx &= \tfrac{1}{6}\int \frac{6}{9+4x^2}\,dx \\ &= \tfrac{1}{6}\int 1\,du \\ &= \tfrac{1}{6}u + c \\ &= \tfrac{1}{6}\arctan\left(\tfrac{2}{3}x\right) + c.\end{aligned}$$

(d) Let $u = e^x$. Then $du/dx = e^x$, so

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx &= \int \frac{1}{\sqrt{1-e^{2x}}}(e^x)\,dx \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \quad (0 < u < 1) \\ &= \arcsin u + c \\ &= \arcsin(e^x) + c.\end{aligned}$$

2.26 **(a)** Here we use integration by substitution. Let $u = 5x^4$. Then $du/dx = 20x^3$, so

$$\begin{aligned}\int x^3\cos(5x^4)\,dx &= \tfrac{1}{20}\int \cos(5x^4)(20x^3)\,dx \\ &= \tfrac{1}{20}\int \cos u\,du \\ &= \tfrac{1}{20}\sin u + c \\ &= \tfrac{1}{20}\sin(5x^4) + c.\end{aligned}$$

(b) Here we use integration by parts.
Let $f(x) = x$ and $g'(x) = \cos(5x)$,
so $f'(x) = 1$ and $g(x) = \frac{1}{5}\sin(5x)$. Then

$$\begin{aligned}\int x\cos(5x)\,dx &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\int \sin(5x)\,dx \\ &= \tfrac{1}{5}x\sin(5x) - \tfrac{1}{5}\left(-\tfrac{1}{5}\cos(5x)\right) + c \\ &= \tfrac{1}{5}x\sin(5x) + \tfrac{1}{25}\cos(5x) + c.\end{aligned}$$

(c) Here neither substitution nor parts is required:

$$\int (x + \cos(5x))\,dx = \tfrac{1}{2}x^2 + \tfrac{1}{5}\sin(5x) + c.$$

(d) Using integration by substitution, let $u = \sqrt{2}x$. Then $du/dx = \sqrt{2}$, so

$$\begin{aligned}\int \frac{1}{1+2x^2}\,dx &= \tfrac{1}{\sqrt{2}}\int \frac{1}{(1+2x^2)}\left(\sqrt{2}\right)\,dx \\ &= \tfrac{1}{\sqrt{2}}\int \frac{1}{1+u^2}\,du \\ &= \tfrac{1}{\sqrt{2}}\arctan u + c \\ &= \tfrac{1}{\sqrt{2}}\arctan\left(\sqrt{2}x\right) + c.\end{aligned}$$

(e) Using integration by substitution, let $u = 7 - x^3$. Then $du/dx = -3x^2$, so

$$\begin{aligned}\int \frac{x^2}{(7-x^3)^7}\,dx &= -\tfrac{1}{3}\int \frac{1}{(7-x^3)^7}(-3x^2)\,dx \\ &= -\tfrac{1}{3}\int \frac{1}{u^7}\,du \\ &= -\tfrac{1}{3}\int u^{-7}\,du \\ &= \tfrac{1}{18}u^{-6} + c \\ &= \tfrac{1}{18}(7-x^3)^{-6} + c.\end{aligned}$$

(f) Using integration by parts,
let $f(x) = x$, and $g'(x) = e^{-\frac{1}{3}x}$,
so $f'(x) = 1$ and $g(x) = -3e^{-\frac{1}{3}x}$. Then

$$\begin{aligned}\int xe^{-\frac{1}{3}x}\,dx &= x\left(-3e^{-\frac{1}{3}x}\right) - \int 1\left(-3e^{-\frac{1}{3}x}\right)\,dx \\ &= -3xe^{-\frac{1}{3}x} + 3\int e^{-\frac{1}{3}x}\,dx \\ &= -3xe^{-\frac{1}{3}x} - 9e^{-\frac{1}{3}x} + c \\ &= -3e^{-\frac{1}{3}x}(x+3) + c.\end{aligned}$$

Section 3

3.1 $$\begin{aligned}(3+\sqrt{-2})(3-\sqrt{-2}) &= 3^2 - (\sqrt{-2})^2 \\ &= 9 - (-2) \\ &= 11.\end{aligned}$$

3.2 **(a)** The solutions are as follows.

$\mathrm{Re}(2i) = \mathrm{Re}(0+2i) = 0$
$\mathrm{Im}(2i) = \mathrm{Im}(0+2i) = 2$
$\mathrm{Re}(-3) = \mathrm{Re}(-3+0i) = -3$
$\mathrm{Im}(-3) = \mathrm{Im}(-3+0i) = 0$

(b) The formula for the solution of a quadratic equation gives

$$\begin{aligned}z &= \frac{6 \pm \sqrt{-64}}{2} \\ &= \frac{6 \pm \sqrt{64}\sqrt{-1}}{2} \\ &= \frac{6 \pm 8i}{2} \\ &= 3 \pm 4i.\end{aligned}$$

So $z_1 = 3 + 4i$, with $\mathrm{Re}(z_1) = 3$ and $\mathrm{Im}(z_1) = 4$, while $z_2 = 3 - 4i$, with $\mathrm{Re}(z_2) = 3$ and $\mathrm{Im}(z_2) = -4$.

3.3 $$\begin{aligned}(-3+4i) + (12-5i) &= (-3+12) + (4-5)i \\ &= 9 - i.\end{aligned}$$

3.4 **(a)** We have

$$\begin{aligned}(-3+2i) \times (6-5i) &= -18 + 15i + 12i - 10i^2 \\ &= -18 + 27i + 10 \\ &= -8 + 27i.\end{aligned}$$

(You would arrive at the same result by substituting in the formula defining a general product.)

(b) We have

$$\begin{aligned}(a+bi) \times (c+di) &= a \times c + (a \times d)i + (b \times c)i \\ &\quad + (b \times d)i^2 \\ &= ac + (ad)i + (bc)i - (bd) \\ &= (ac - bd) + (ad + bc)i,\end{aligned}$$

as is given in the definition of multiplication.

3.5 $$\begin{aligned}z - w &= (-3+4i) - (12-5i) \\ &= (-3-12) + (4+5)i \\ &= -15 + 9i.\end{aligned}$$

3.6 We can write the product $z \times w$ as

$$(a+bi) \times \left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right)$$

and use the formula for the multiplication of complex numbers. Thus

$$\begin{aligned}z \times w &= \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}\right) + \left(\frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right)i \\ &= 1 + 0i \\ &= 1.\end{aligned}$$

3.7 From the definition of the reciprocal,

$$z^{-1} = \frac{12+5i}{12^2+5^2} = \frac{12+5i}{169}.$$

If $u \times z = w$, then

$$\begin{aligned}u = \frac{w}{z} = w \times z^{-1} &= (-3+4i) \times \frac{12+5i}{169} \\ &= \frac{-56+33i}{169} \\ &= \frac{-56}{169} + \frac{33}{169}i.\end{aligned}$$

3.8 **(a)** $$\begin{aligned}z \times \overline{z} &= (a+bi) \times (a-bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2.\end{aligned}$$

(b) Multiply top and bottom of the fraction by the complex conjugates of both $1+i$ and $2-3i$.

$$\begin{aligned}&\frac{1+2i}{(1+i) \times (2-3i)} \\ &\quad = \frac{(1+2i) \times (1-i) \times (2+3i)}{(1+i) \times (1-i) \times (2-3i) \times (2+3i)}.\end{aligned}$$

Keeping the pairs of complex conjugates together in the denominator helps with the simplification, since the terms in i then cancel. In the denominator we have

$(1+i) \times (1-i) = 1 - i^2 = 1 + 1 = 2,$
$(2-3i) \times (2+3i) = 2^2 - (3i)^2 = 4 + 9 = 13.$

The numerator is

$$\begin{aligned}(1+2i) \times (1-i) \times (2+3i) &= (3+i) \times (2+3i) \\ &= 3 + 11i.\end{aligned}$$

So the fraction simplifies to

$$\frac{3+11i}{2 \times 13} = \frac{3+11i}{26}.$$

(c) The formula for the solution of a quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 \geq 4ac$, the solutions are real.

If $b^2 < 4ac$,

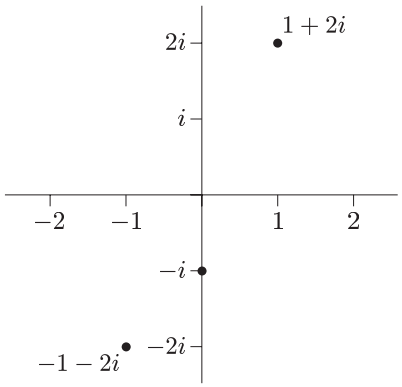
$$b^2 - 4ac = (-1)(4ac - b^2),$$

where $4ac - b^2 > 0$, so

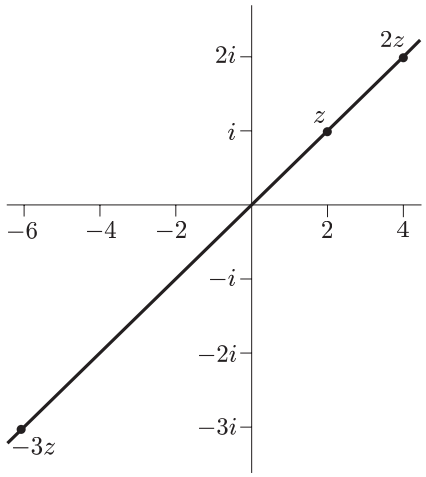
$$x = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} = \frac{-b \pm \left(\sqrt{4ac - b^2}\right) i}{2a}.$$

These solutions are complex conjugates, as they are of the form $x_1 = m + ni$ and $x_2 = m - ni = \overline{x}_1$, where m and n are real numbers.

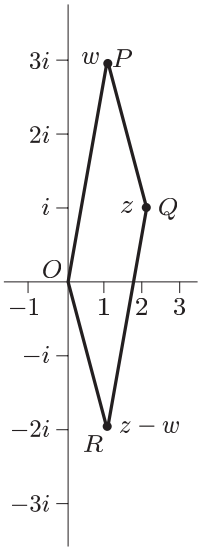
3.9



3.10 **(a)** All real multiples of z lie on the straight line passing through z and the origin.



(b) We have $z - w = (2 + i) - (1 + 3i) = 1 - 2i$, which is represented by $(1, -2)$ on the Argand diagram below. $OPQR$ is a parallelogram, whose corners are points representing the complex numbers 0, w, z and $z - w$.

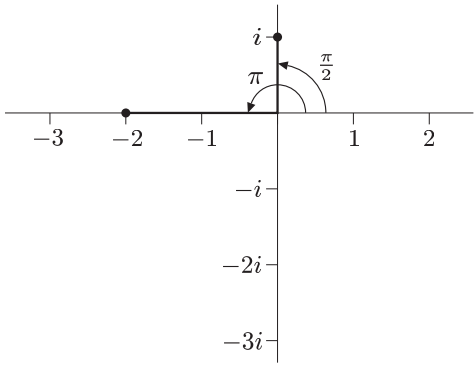


3.11 **(a)** For $\langle 1, 0\rangle$, we have the Cartesian form $1\cos 0 + (1\sin 0)i = 1 + 0i = 1$.

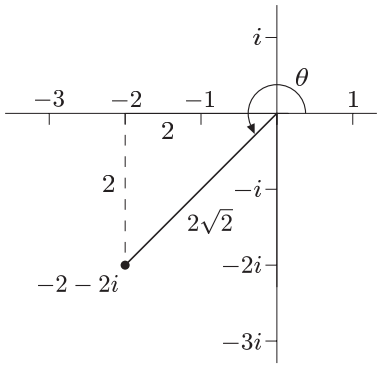
(b) For $\left\langle 1, \frac{\pi}{4}\right\rangle$, we have the Cartesian form $1\cos\frac{\pi}{4} + \left(1\sin\frac{\pi}{4}\right) i = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

(c) For $\left\langle 2, \frac{\pi}{2}\right\rangle$, we have the Cartesian form $2\cos\frac{\pi}{2} + \left(2\sin\frac{\pi}{2}\right) i = 0 + 2i = 2i$.

3.12 **(a)** and **(b)** The complex numbers i and -2 are shown on the Argand diagram below. Their polar forms are $\langle 1, \frac{\pi}{2}\rangle$ and $\langle 2, \pi\rangle$, respectively.



(c) Here $r^2 = (-2)^2 + (-2)^2$ and hence $r = 2\sqrt{2}$. To find the argument, we need to locate the quadrant in which z lies. An Argand diagram helps.



This shows that z lies in the third quadrant. We can also see that the angle θ has

$$\cos\theta = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}, \quad \sin\theta = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

So we have

$$\theta = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{5\pi}{4}.$$

3.13 **(a)** Using the polar forms of z and w,

$$\begin{aligned} z \times w &= \left\langle \sqrt{2} \times 2\sqrt{2}, \tfrac{3\pi}{4} + -\tfrac{3\pi}{4} \right\rangle \\ &= \langle 4, 0 \rangle. \end{aligned}$$

In Cartesian form, this is 4.

(b) Using the polar forms of z and w,

$$\begin{aligned} z \times z &= \left\langle \sqrt{2} \times \sqrt{2}, \tfrac{3\pi}{4} + \tfrac{3\pi}{4} \right\rangle \\ &= \left\langle 2, \tfrac{3\pi}{2} \right\rangle \\ &= \left\langle 2, -\tfrac{\pi}{2} \right\rangle. \end{aligned}$$

In Cartesian form, this is

$$2\left(\cos\left(-\tfrac{\pi}{2}\right) + i\sin\left(-\tfrac{\pi}{2}\right)\right) = -2i.$$

(c) Using the polar form of z,

$$\begin{aligned} z \times (z \times z) &= \left\langle \sqrt{2} \times 2, \tfrac{3\pi}{4} - \tfrac{\pi}{2} \right\rangle \\ &= \left\langle 2\sqrt{2}, \tfrac{\pi}{4} \right\rangle. \end{aligned}$$

In Cartesian form, this is

$$2\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right) = 2 + 2i.$$

3.14 Using (3.5),

$$\begin{aligned} e^{2+3i} &= e^2(\cos 3 + i\sin 3) \\ &\simeq -7.315 + 1.043i. \end{aligned}$$

3.15 **(a)** In polar form,

$$e^z = \langle e^a, b \rangle, \quad e^w = \langle e^c, d \rangle.$$

(i) Using the rule for multiplication of complex numbers in polar form,

$$\begin{aligned} e^z \times e^w &= \langle e^a \times e^c, b + d \rangle \\ &= \langle e^{a+c}, b + d \rangle. \end{aligned}$$

(Since a and c are real numbers, $e^a \times e^c = e^{a+c}$.)

(ii) We have $z + w = (a + c) + (b + d)i$. So, in polar form,

$$e^{z+w} = \langle e^{a+c}, b + d \rangle.$$

So we do have, for complex exponentials,

$$e^z \times e^w = e^{z+w}.$$

(b) We have $-z = -a - bi$. So, in polar form,

$$e^{-z} = \langle e^{-a}, -b \rangle.$$

Using the rule for complex multiplication in polar form,

$$\begin{aligned} e^z \times e^{-z} &= \langle e^a \times e^{-a}, b + (-b) \rangle \\ &= \langle e^{a-a}, 0 \rangle \\ &= \langle e^0, 0 \rangle \\ &= \langle 1, 0 \rangle, \end{aligned}$$

which is the polar form of 1.

3.16 The complex number $1 + i$ has modulus $\sqrt{2}$ and argument $\frac{\pi}{4}$, and hence has polar form $\left\langle \sqrt{2}, \frac{\pi}{4} \right\rangle$. This transforms directly into the exponential form $\sqrt{2}e^{\frac{\pi}{4}i}$. That is,

$$1 + i = \left\langle \sqrt{2}, \tfrac{\pi}{4} \right\rangle = \sqrt{2}e^{\frac{\pi}{4}i}.$$

3.17 **(a)** $3 - i$.

(b) $-1 + 2i$.

(c) $1 + 3i$.

(d) $-4 - 2i$.

(e) $4 - 3i$.

(f) $\dfrac{1 + 2i}{5}$.

(g) $\dfrac{2 + i}{1 - 2i} = \dfrac{(2 + i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \dfrac{5i}{5} = i.$

3.18 $\overline{z} = a - bi$ and $\overline{w} = c - di$. So

$$\overline{z} + \overline{w} = (a + c) - (b + d)i.$$

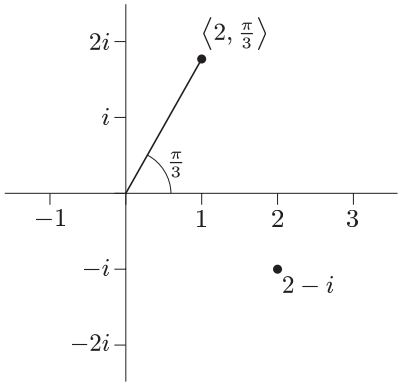
Now $z + w = (a + c) + (b + d)i$. So

$$\overline{z + w} = (a + c) - (b + d)i.$$

So we do have, as required,

$$\overline{z} + \overline{w} = \overline{z + w}.$$

3.19 **(a)**



(b) $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\arccos\frac{2}{\sqrt{5}} \simeq 0.4636$. Since z lies in the fourth quadrant, $\arg(z) \simeq -0.4636$. We have the polar form

$$z = \left\langle \sqrt{5}, -0.4636 \right\rangle.$$

The Cartesian form of w is

$$w = 2\cos\tfrac{\pi}{3} + 2i\sin\tfrac{\pi}{3} = 1 + i\sqrt{3}.$$

(c) Using the polar form of w,

$$w^3 = \left\langle 2, \tfrac{\pi}{3} \right\rangle^3 = \left\langle 2^3, \tfrac{3\pi}{3} \right\rangle = \langle 8, \pi \rangle = -8.$$

3.20 **(a)** With $z = 12 - 5i$, we obtain the following.

(i) $\overline{z} = 12 + 5i$.

(ii) $|z| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$.

(iii) $z \times \overline{z} = 12^2 + 5^2 = 169$.

Notice that $|z|^2 = z \times \overline{z}$.

(b) If $z = a + bi$, then $\overline{z} = a - bi$ and

$$z \times \overline{z} = (a + bi) \times (a - bi) = a^2 + b^2 = |z|^2.$$

(c) If we multiply both the top and the bottom of the fraction $1/z$ by $\overline{z}$ (as we would do to simplify this fraction for a particular complex number z), we have

$$\frac{1}{z} = \frac{\overline{z}}{z \times \overline{z}} = \frac{\overline{z}}{|z|^2},$$

using the result of (b).

3.21 From the definition, we have

$$\begin{aligned} e^{-1+i\pi/3} &= e^{-1}(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}) \\ &= e^{-1}\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) \\ &\simeq 0.184 + 0.319i. \end{aligned}$$

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